to the author to be eminently desirable in this country for furthering such a work in an efficient and collaborative way. It is a matter of sincere congratulation that the Council of Scientific and Industrial Research of the Government of India has seen fit to make a grant for the development of the physical methods to the determination of the geological age. A great deal of work has to be done in this connection and co-ordination of efforts between the geologists and physicists has to be achieved. Some of the fundamental apparatus for routine determination, e.g. mass-spectrograph for isotope ratios, are still unavailable in this country.

It is hoped that with future co-operative efforts greater strides in the path of progress will be made in this field of fundamental work.

We wish to express our heartfelt thanks to Prof. D. N. Wadia for having gone through the paper, and helped us with very valuable suggestions in the preparation of the introductory parts.

References

Bower (1939). Radio-activity of Potassium and Geologic Time. *Science*, **86**, 198.

Daly. Igneous Rock.

Ellsworth, H. (1925). Radioactive minerals as Geologic Indicators. *Amer. Journ., Sc.* **9**, 127.

Evans, R. D. (1935). Radio-activity of Solids. *Phys. Rev.*, **48**, 503.

Harker. Metamorphosis.

Henderson, H. G. *et al* (1937). Studies of Pleochroic haloes. *Proc. Roy. Soc.*, **158**, 199.

Holmes. Physical Geology.

Holmes. The Age of the Earth.

Hovell, Lane *et al.* (1941). Symposia, Terrestrial Radio-activity *Journ. Ap. Phy.*, **12**, 296.

Keevil, N. B. (1939). The calculation of Geologic Age. *Amer. J. Sc.* **237**, 195.

——(1938). Economic Geology.

Lane, A. C. (1941). The Earth shows its Age. *Science Progress.*

Lane, A. C. and Urry (1936). Estimation of Geologic Age. *Journ. Ch. Phy.*, **4**, 34.

Nag *et al.* (1944). Radio-active Content of Indian Minerals. *Proc. Nat. Inst. Sc. Ind.*, **10**, 167.

National Research Council Bull. Common Geologic time annual reports, 1931-34.

Nier, A. (1941). Isotopic Constitution of Pb and C. *Phys. Rev.*, **60**, 112.

Rose, M. *et al.* (1936). Isotopic const. of Pb from H.F.S. *Phys. Rev.*, **50**, 792.

Sahni, B. (1944). Tertiary Fossils in Salt Range. *Nature*, **153**, 462.

Sarkar, P. B. (1941). On Age of some Radio-active Minerals. *Science and Culture*, **7**, 119.

Sharma, S. (1943). Classical List of Indian Minerals. *Tr. In. Ce. Soc.*, **3**, 90.

Urry (1939). Ages by He-Method. *Bull. Geo. Soc. Am.*, **47**, 1217.

Wadia, D. N. Geology of India.

82. ON THE PROPAGATION OF ELECTRO-MAGNETIC WAVES THROUGH THE UPPER ATMOSPHERE

M. N. Saha*, B. K. Banerjea and U. C. Guha

(*Ind. Jour. Phys.*, **21**, 181, 1947)

Abstract

This paper reports a comprehensive working of the problems of an ionised atmosphere, traversed by a magnetic field, as in the case of the Earth's atmosphere. Expressions are deduced for electrical polarisation and complex conductivity for such an atmosphere when traversed by radio waves, in a tensor-form, as first suggested by Darwin. The equations of propagation of radio frequency waves through such a medium are obtained by the use of cardinal axes, and then the equations of vertical propagation are deduced. Expressions are obtained for refractive indices of ordinary and extraordinary waves, which agree with the expressions given by Appleton. Expressions are obtained for polarisation, absorption etc. of the radio waves travelling in the ionosphere. Curves are given for the polarisation ratio and refractive indices of the two waves as functions of the magnetic latitude of the place of observation.

Introduction

Ever since the classical works of Appleton (1932) and Hartree (1932), the problem of the propagation of e.m. waves in the ionosphere has received attention from numerous workers. Summaries of these works are available in various reports. Recently B. K. Banerjea (1947) made a critical and comparative study of the fundamental methods of Appleton (1932), Hartree (1932), Saha, Rai and Mathur (1937) and Saha and Banerjea (1945) and showed that these various methods can be deduced as special cases of a general method developed according to Darwin's (1925) suggestion of treating the e.m. properties of the medium as tensor quantities. The present paper continues the treatment further and aims at giving a true wave formulation of the general problem. For the convenience of the reader some results of the previous works carried out by the senior author and his early collaborators are included so that no further references to these papers are needed. Part

*Fellow of the Indian Physical Society.

of the results mentioned in the earlier parts are not new, but have been derived in a novel, easier and unitary way.

The Displacement of the Ions in the Ionosphere

The equation of motion of the charged ions referred to any system of co-ordinates can be written as:

$$\frac{d^2\rho}{dt^2} + \nu\frac{d\rho}{dt} + \frac{e}{mc}\left[\mathrm{H}\times\frac{d\rho}{dt}\right] = \frac{e}{m}E, \qquad .. \quad (1.1)$$

where ρ = displacement vector with components (ξ,η,ζ).

e, m = magnitude of the charge and mass of the ion respectively.

ν = collision frequency of the ions.

H = Earth's magnetic field.

$E = E_0\cos pt$, electric vector of the incident electromagnetic wave.

The effect of the magnetic vector and the space charges have been omitted as usual. The notation conforms as closely as possible to those used by Appleton (1932) and Saha, Rai and Mathur (1937) and B. K. Banerjea (1947).

It can easily be verified that the solution of the above equation with $E=E_0\cos pt$ is the real part of the solution obtained with $E=E_0e^{ipt}$; we use E in this latter form because solution is then easy to obtain. The quantity analogous to the static conductivity now comes out as complex (Stratton, 1942), whose real part gives ordinary refractive index and the imaginary part gives deviation of the refractive index from unity.

Introducing the polarisation vector $P=4\pi Ne\rho$ where N is the ion-concentration and using the abbreviations,

$$\frac{mp^2}{Ne^2} = \frac{4\pi p^2}{p_0^2} = \frac{4\pi}{r}, \quad p_0^2 = \frac{4\pi Ne^2}{m}, \quad r = \frac{p_0^2}{p^2},$$

$$\nu/p = \delta, \qquad \mathrm{I} - i\delta = \beta, \qquad .. \quad (1\cdot2)$$

$$\frac{e\mathbf{H}}{mc} = \mathbf{p}_h, \quad \frac{\mathbf{p}_h}{p} = \omega \text{ with components } \omega_x, \omega_y, \omega_z.$$

We get from equation (1·1) replacing $\rho\ (\xi, \eta, \zeta)$ by $\frac{1}{4\pi Ne}(P_x, P_y, P_z)$

$$\left.\begin{aligned} \beta P_x + i\omega_z P_y - i\omega_y P_z &= -E_x \\ -i\omega_z P_x + \beta P_y + i\omega_x P_z &= -E_y \\ i\omega_y P_x - i\omega_x P_y + \beta P_z &= -E_z \end{aligned}\right\}. \qquad .. \quad (1\cdot3)$$

The solution of these equations can be briefly written as

$$P = \mathrm{A}\,\triangle.E, \qquad .. \quad (1\cdot4)$$

where $\mathrm{A} = \frac{r}{\beta(\beta^2-\omega^2)}$ and $\triangle$ is a tensor given by the matrix,

$$\triangle = \begin{vmatrix} \omega_x^2-\beta^2 & \omega_x\omega_y+i\beta\omega_z & \omega_x\omega_z-i\beta\omega_y \\ \omega_y\omega_x-i\beta\omega_z & \omega_y^2-\beta^2 & \omega_y\omega_z+i\beta\omega_x \\ \omega_z\omega_x+i\beta\omega_y & \omega_z\omega_y-i\beta\omega_x & \omega_z^2-\beta^2 \end{vmatrix} \qquad (1.5)$$

It has been shown by Saha and Banerjea (1945) that the tensor possesses certain "Cardinal Axes" which may be denoted by 1, 2, 3. "1" is the direction of the earth's magnetic field, "2" is the line perpendicular to the magnetic meridian, and "3" is the line perpendicular to "1" lying in the magnetic meridian. The relation between these axes and the axes commonly used in ionospheric problems with XZ as magnetic meridian and OZ as vertical is shown in the diagram below:

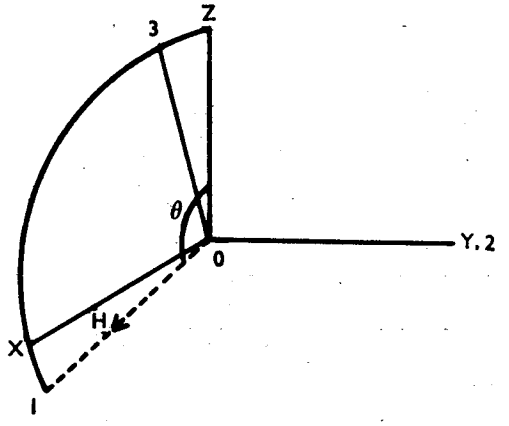

Fig. 1
Shows disposition of cardinal axes (1, 2, 3) with axes used generally in considering vertical propagation.

In this figure $\theta = \angle ZO1$ is called the angle of propagation. The axis O1 is always along the positive direction of H. In general literature on ionospheric problems, the positive direction of H is generally not expressed quite clearly, with the result that the sense of rotation of the electric and magnetic vectors of the returning radio wave is left unclarified. In what follows the positive direction of H is along the positive direction of the magnetic lines of force, *i.e.* in the northern hemisphere it is downward and in the southern the reverse is the case.

Choice of these axes is equivalent to putting

$$w_1 = w, \quad w_2 = w_3 = 0,$$

where w_1, w_2, w_3 are the components of w along (1, 2, 3) axes. We have then

$$\triangle = -\begin{vmatrix} \beta^2-\omega^2 & 0 & 0 \\ 0 & \beta^2 & i\beta\omega \\ 0 & -i\beta\omega & \beta^2 \end{vmatrix} \qquad .. \quad (1.6)$$

The complex conductivity σ of the medium, defined by the equation

$$\sigma\ .E = \text{current} = \frac{1}{l}\mathrm{N}e\frac{d\rho}{dt} = -ip\mathrm{N}e\rho = -\frac{ip}{4\pi}P = -\frac{ip\mathrm{A}}{4\pi}\triangle.E$$

is a tensor quantity defined by the matrix,

$$\sigma = \frac{iNe^2}{m}\begin{vmatrix} \frac{1}{p-i\nu} & o & o \\ o & \frac{p-i\nu}{(p-i\nu)^2-p^2h} & \frac{-iph}{(p-i\nu)^2-p^2h} \\ o & \frac{iph}{(p-i\nu)^2-p^2h} & \frac{p-i\nu}{(p-i\nu)^2-p^2h} \end{vmatrix} \quad (1.7)$$

The steady current conductivity σ_* is obtained from above by putting $p=o$. We have

$$\sigma_* = -\frac{Ne^2}{m}\begin{vmatrix} \frac{1}{\nu} & o & o \\ o & \frac{\nu}{p_h^2+\nu^2} & \frac{p_n}{p_h^2+\nu^2} \\ o & \frac{-p_n}{p_h^2+\nu^2} & \frac{\nu}{p_h^2+\nu^2} \end{vmatrix} \quad .. \quad (1.8)$$

Thus in the direction of the magnetic field, the steady current conductivity is $\frac{Ne^2}{m\nu}$. We have the components of current as

$$\left.\begin{aligned} i_1 &= \frac{Ne^2}{m\nu}E_1 \\ i_2 &= \frac{Ne^2}{m(p_h^2+\nu^2)}(\nu E_2+p_hE_3) \\ i_3 &= \frac{Ne^2}{m(p^2h+\nu^2)}(-p_hE_2+\nu E_3) \end{aligned}\right\} \quad .. \quad (1.9)$$

If $E_3=o$, we have $i_2=\frac{Ne^2}{m(p_h^2+\nu^2)}\nu E_2$. The quantity $\frac{Ne^2\nu}{m(p_h^2+\nu^2)}$ is known as transverse conductivity. We have besides, the current $i_3=-\frac{Ne^2ph}{m(p_h^2+\nu^2)}E_2$ along the Z axis, though there may be no e.m.f. in that direction.

The Polarisation Vector—The polarisation vector P is defined as $P=\frac{4\pi i}{p}\sigma.E$ and we can easily deduce that

$$E_1=-\frac{\beta}{r}P_1,\quad E_2\pm iE_3=-\frac{\beta\mp\omega}{r}(P_2\pm iP_3) \quad .. \quad (1\cdot10)$$

The Electric Displacement Vector and the Complex Dielectric Tensor—The electric displacement vector $D=E+P$ may be expressed as $D=\mathbf{K}.E.$, where $\mathbf{K}$ is the complex dielectric tensor given by the matrix,

$$\mathbf{K}=\begin{vmatrix} 1-r/\beta & o & o \\ o & 1-r\beta/(\beta^2-\omega^2) & ir\omega/(\beta^2-\omega^2) \\ o & -ir\omega/(\beta^2-\omega^2) & 1-r\beta/(\beta^2-\omega^2) \end{vmatrix} \quad .. \quad (1\cdot11)$$

2. The Maxwellian Equations

From the Maxwellian equations:

$$\nabla\times H=\frac{1}{c}\frac{\partial D}{\partial t},\ \nabla\times E=-\frac{1}{c}\frac{\partial H}{\partial t},\ \nabla.D=\nabla.H=0 \quad .. \quad (2\cdot1)$$

We get by the usual methods, the equations of propagation for the electric and magnetic vectors in the form:

$$\left.\begin{aligned} &\triangle^2E_1+\frac{p^2}{c^2}(1-r/\beta)E_1=o \\ &\nabla^2(E_2\pm iE_3)+\frac{p^2}{c^2}\left(1-r/(\beta\pm\omega)\right)(E_2\pm iE_3)=o \\ &\nabla^2H+\frac{p^2}{c^2}H=-\frac{4\pi}{c}\nabla\times(\sigma.E) \end{aligned}\right\} \quad (2\cdot2)$$

The Wave Equations for Vertical Propagation in any Latitude:—Let us first confine ourselves to the propagation along the vertical Z-axis, so that ∇ and ∇^2 simply reduce to $\frac{d}{dz}$ and $\frac{d^2}{dz^2}$. Introducing the new variable $u=pz/c$, we get from (2·2)

$$\frac{d^2E_1}{du^2}+\left(1-\frac{r}{\beta}\right)E_1=o. \quad .. \quad (2\cdot3a)$$

$$\frac{d^2}{du^2}(E_2+iE_3)+\left(1-\frac{r}{\beta-\omega}\right)(E_2+iE_3)=o. \quad .. \quad (2\cdot3b)$$

$$\frac{d^2}{du^2}(E_2-iE_3)+\left(1-\frac{r}{\beta+\omega}\right)(E_2-iE_3)=o. \quad .. \quad (2\cdot3c)$$

The components of the vector E in two systems (1, 2, 3) and with XZ-plane as magnetic meridian and OZ as vertical are related as:

$$\left.\begin{aligned} &E_1=E_x\sin\theta+E_z\cos\theta; && E_x=E_1\sin\theta-E_3\cos\theta. \\ &E_2=E_y; && E_y=E_z. \\ &E_3=-E_x\cos\theta+E_z\sin\theta; && E_z=E_1\cos\theta+E_3\sin\theta. \\ &\omega_x=\omega\sin\theta, \quad \omega_y=o, && \omega_z=-\omega\cos\theta \end{aligned}\right\} \quad (2\cdot4)$$

The equations (2.3) as such are not suitable for use when we consider the propagation of plane waves, for such cases we have to use in conjunction with (2.2), the Maxwellian condition $\nabla.D=o$. For vertical propagation, this reduces to $\frac{d}{dz}D_z=o$, *i.e.*, $D_z=o$, since the steady components of D, if any, are unimportant in the study of the wave propagation:

From $D_z=o$ and (1.4) and (1.5) we have eliminating P_x, P_y, P_z and putting $w_y=o$.

$$E_z=\frac{rw_x}{c'}(-w_zE_x+i\beta E_y), \quad .. \quad (2.5)$$

where $C'=\beta(\beta^2-w^2)-r(\beta^2-w_z^2)$.

Multiplying (2.3a) by sin θ and the difference of (2.3b) and (2.3c) by $-\cos\theta$, adding the results and then replacing

E_1, E_2, E_3 by their equivalent expressions in terms of E_x, E_y, E_z from (2.4) we get after some simplification,

$$\frac{d^2E_x}{du^2} + K_1E_x - iLE_y = o, \quad .. \quad (2.6)$$

where
$$\left.\begin{aligned} K_1 &= 1 - r\frac{\beta^2 - r\beta - \omega^2\sin^2\theta}{C'} \\ L &= \frac{-r(\beta - r)\,\omega\cos\theta}{C'} \end{aligned}\right\}. \quad .. \quad (2.7)$$

Again replacing E_2 and E_3 by E_x, E_y, E_z in equation (2.3b) from (2.4) and E_z by E_x and E_y from (2.2), we have after some work,

$$\frac{d^2E_y}{du^2} + K_2E_y + iLE_x = o, \quad .. \quad (2.8)$$

where
$$K_2 = 1 - r\frac{\beta^2 - r\beta}{C'}, \quad .. \quad (2.9)$$

Equation (2.6) and (2.8) were obtained explicitly in this form by Saha, Rai and Mathur (1937). Equivalent equations with vector components of the ordinary and extraordinary waves intermixed in each equation were obtained by Rydbeck (1944). But equations in this form do not help much in the understanding of the phenomena, unless the coupling term L between the variables vanishes. This takes place at $\theta = \pi/2$, *i.e.*, at the magnetic equator, where the equations of propagation become,

$$\left.\begin{aligned} \frac{d^2E_x}{du^2} + \left(1 - \frac{r}{\beta}\right)E_x &= o \\ \frac{d^2E_y}{du^2} + \left(1 - \frac{r}{\beta - \dfrac{\omega}{\beta - r}}\right)E_y &= o \end{aligned}\right\}. \quad (2.10)$$

For the magnetic poles, $\theta = \pi$ and o, and for these values of θ, $K_1 = K_2$; for $\theta = \pi$, *i.e.*, mag N-pole, the equation of propagation takes the form:

$$\frac{d^2}{du^2}(E_x \pm iE_y) + \left(1 - \frac{r}{\beta \mp \omega}\right)(E_x \pm iE_y) = o. \quad .. \quad (2.11)$$

For $\theta = o$, *i.e.*, mag. S-Pole, the equation similarly reduces to

$$\frac{d^2}{du^2}(E_x \pm iE_y) + \left(1 - \frac{r}{\beta \pm \omega}\right)(E_x \pm iE_y) = o, \quad .. \quad (2.11a)$$

Equation in these forms were studied by Saha and Rai (1937), for the case when damping is negligible, *i.e.*, $\beta = 1$, from the wave mechanical point of view. The Chapman layer of ion-distribution was treated as a potential barrier and the penetration of the waves under certain simplifying assumptions were studied in the same way as Gamow did in his famous work on the "Penetration of the Potential Barrier of Nuclei of Atoms by High Energy particles."

Recently Rydbeck (1942) has studied these equations when the coupling term L vanishes; he has given an elaborate treatment of the wave equations for magnetic equator and taking a parabolic ion-layer and using Weber's parabolic functions he has obtained expressions for the reflection co-efficient, transmission co-efficient and phase retardation of the wave in a thin friction free parabolic layer. In the ray treatment of Appleton we practically confine ourselves to these two limiting cases, *viz.*, their quasi-longitudinal case is for $\theta = \pi$, o, *i.e.*, $K_1 = K_2$ and their quasi-transverse case, *i.e.*, $\theta = \frac{\pi}{2}$, $L = o$.

The following method will be found applicable to all stations. Multiplying both sides of (2.8) by "iF," and adding to (2.6), where F is an indeterminate multiplier to be presently, determined, we have

$$\frac{d^2E_x}{du^2} + iF\frac{d^2E_y}{du^2} + (K_1 - FL)E_x + \left(K_2 - \frac{L}{F}\right)iFE_x = o. \quad (2.12)$$

Now choose F in such a way that $K_1 - FL = K_2 - \frac{L}{F}$ so that F is given by the equation

$$F^2 - \frac{K_1 - K_2}{L}F - 1 = o. \quad .. \quad (2.13)$$

Put
$$\frac{K_1 - K_2}{L} = \frac{\omega\sin^2\theta}{(r-\beta)\cos\theta} = 2G = 2g\cos\alpha\, e^{i\alpha},$$

where
$$g = \frac{\omega\sin^2\theta}{(r-1)\cos}, \quad \tan\alpha = \frac{\delta}{1-r}. \quad .. \quad (2.14)$$

Let F_1, F_2 be the roots of equation (2.13). Then

$$F_1, F_2 = G \pm \sqrt{+G^2}$$
$$= g \pm \sqrt{1+g^2} \quad \text{for } \delta = o. \quad .. \quad (2.15)$$

Now turning to equation (2.12) we can rewrite it in the form

$$\frac{d^2}{du^2}(E_x + iFE_y) + q^2(E_x + iFE_y) - 2i\frac{dF}{du}\frac{dE_y}{du} - i\frac{d^2F}{du^2}E_y = o \quad .. \quad (2.16)$$

where q has the two values given by

$$q_1^2 = K_2 - \frac{L}{F_1} = K_2 + LF_2 = 1 - \frac{r}{c'}(\beta - \gamma)(\beta + \omega\cos\theta\, F_2)$$
$$= 1 - \frac{r}{\beta + \omega\cos\theta\, F_1}. \quad .. \quad (2.17)$$

$$q_2^2 = K_1 - \frac{L}{F_2} = K_1 + LF_1 = 1 - \frac{r}{c'}(\beta - r)(\beta + \omega\cos\theta\, F_1)$$
$$= 1 - \frac{r}{\beta + \omega\cos\theta\, F_2}, \quad .. \quad (2.18)$$

for $C' = (\beta - r)(\beta + \omega\cos\theta\, F_1)(\beta + \omega\cos\theta\, F_2)$.

In those cases where the quantities $\frac{dF}{du}$, $\frac{d^2F}{du^2}$ can be neglected, the equations can be written as

$$\left.\begin{aligned}\frac{d^2}{du^2}(E_x+iF_1E_y)+q_1^2(E_x+iF_1E_y)=0\\ \frac{d^2}{du^2}(E_x+iF_2E_y)+q_2^2(E_x+iF_2E_y)=0\end{aligned}\right\}. \quad .. \quad (2.16a)$$

These signify that the beam is broken up into two, with the refractive indices q_1, and q_2, and polarisations determined by F_1 and F_2 (*Vide* §4).

We next proceed to discuss the case of friction free atmosphere. In this case we have

$$q_1^2 = 1 - \frac{r}{1+\omega\cos\theta\, F_1} \quad .. \quad (2.17a)$$

$$q_2^2 = 1 - \frac{r}{1+\omega\cos\theta\, F_2} \quad .. \quad (2.18a)$$

Both q_1, q_2 are to be continuous functions of r. We find from the expression for g, that for $r\to 1$, $g\to\infty$. At this point, q_1^2, q_2^2 should obey the condition of continuity, *i.e.*,

$$\underset{r=1-0}{Lt}\,(q_1^2, q_2^2) = \underset{r=1+0}{Lt}\,(q_1^2, q_2^2).$$

Taking first q_1, we find that if we take for the region $r=0$ to $r=1$

$$F_1=g-\sqrt{1+g^2}, \text{ consequently } q_1^2=1-\frac{r}{1-\omega\cos\theta(\sqrt{1+g^2}-g)} \quad (2.17b)$$

Then q_1 varies from 1 to 0 in the domain $r=0$ to 1. As the value is to be continuous, and since g on crossing over to $r=1+0$, becomes negative, we find that for this region ($r>0$) we should put

$$F_1 = \sqrt{1+g^2} - |g|$$

$$i.e., \; q_1^2 = 1 - \frac{r}{1+\omega\cos\theta\,(\sqrt{1+g^2}-|g|)}. \quad .. \quad (2.17c)$$

These expressions for q_1^2 has no singularity at any point and it is identical with the expression for the refractive index of the ordinary wave as given by Appleton. A($r-q_1^2$) curve for different values of θ from expressions (2.17 *b*, *c*) is given in Fig. 2, for $\omega<1$, and $\omega>1$.

For the other beam we can now substitute the corresponding value of F_2, we obtain:

$$q_2^2=1-\frac{r}{1+\omega\cos\theta\, F_2}=1-\frac{r}{1+\omega\cos\theta\,(\sqrt{1+g^2}+g)}$$

for $r\gtrless 1$ .. (2.18*b*)

$$=1-\frac{r}{1-\omega\cos\theta\,(\sqrt{1+g^2}+|g|}\text{ for } r>1. \quad .. \quad (2.18c)$$

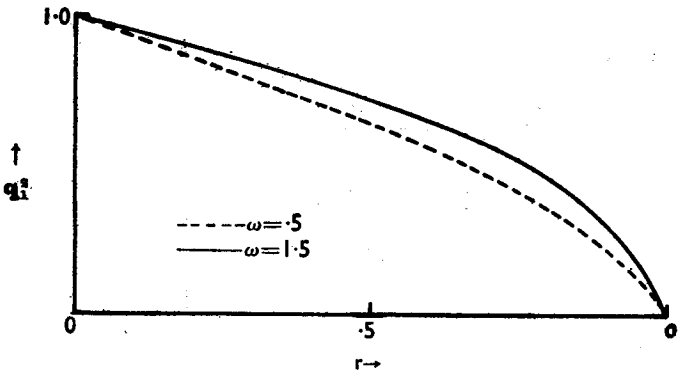

Fig. 2

Variation of the square of the refractive index for the o-wave with electron concentration $r=\left(\frac{4\pi e^2}{mp^2}\,N\right)$, for $p_h/p=\omega<1$ (here ·5) and $\omega>1$ (here 1.5)

It can be easily shown that for $\omega<1$, $(r-q_2^2)$—curve starts from (o,1) passes through $(1-\omega,\ o)$ and a point of infinite singularity at $r=\frac{1-\omega^2\cos^2\theta}{1-\omega^2}$ where it passes from $-\infty$ to $+\infty$, passes through the point (1, 1) and $(1+\omega,\ o)$ for all values of θ.

$q^2{}_2$ has therefore to be identified with the square of the refractive index of the extraordinary wave (Fig. 3).

For $\omega>1$, we find that the curve passes through (o, 1) and (1,1), between $r=0$, and 1, $q^2{}_2>1$ but after (1,1) the value of $q^2{}_2$ becomes less than unity and gradually tends to the value zero at $r=1+\omega$, after which it is negative (Fig. 3).

The quantities $g(\omega,\ r,\ \theta)$, $-F=\sqrt{1+g^2}-|g|$ which occur in this work are functions of ω, r and θ.

In Table 1, the function $g(\omega,\ o,\theta)$ has been given for various values of ω and θ. To obtain $g\,(\omega, r, \theta)$ we have to divide $g(\omega, o, \theta)$ by $(r-1)$.

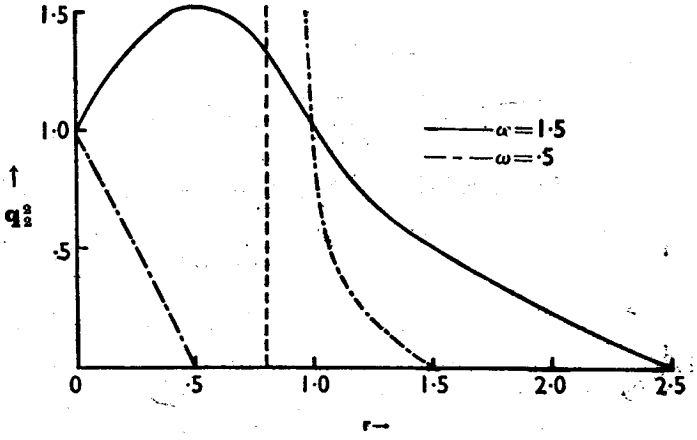

Fig. 3

Variation of the square of the refractive index for the *e*- wave with electron concentration $\left(r=\frac{4\pi e^2}{mp^2}\,N\right)$ for $p_h/p=\omega<1$ (here ·5) and $p_h/p=\omega>1$ (here 1·5).

TABLE 1

$g=\omega\cos^2\theta/2\sin\theta$, $r=0$

$\theta°$	$\omega=\cdot1$	$\omega=\cdot2$	$\omega=\cdot3$	$\omega=\cdot5$	$\omega=\cdot8$	$\omega=1\cdot0$	$\omega=1\cdot5$	$\omega=2$	$\omega=5$	$\omega=10$	$\omega=20$	$\omega=50$	$\omega=100$
91°	2·8640	5·7270	8·591	14·315	22·908	28·635	42·955	57·270	143·15	286·35	572·70	1431·5	2863·5
92°	1·4310	2·8620	4·293	7·155	11·448	14·310	21·465	28·620	71·55	143·10	286·20	715·5	1431·0
93°	·9525	1·9050	2·858	4·763	7·620	9·525	14·290	19·050	47·63	95·25	190·50	476·3	952·5
94°	·7135	1·4270	2·141	3·568	5·708	7·135	10·705	14·270	35·68	71·35	142·70	356·3	713·5
95°	·5695	1·1390	1·709	2·848	4·556	5·695	8·545	11·390	28·48	56·95	113·90	284·8	569·5
190°	·2793	·5586	·838	1·397	2·234	2·793	4·190	5·586	13·97	27·93	55·86	139·7	279·3
105°	·1802	·3604	·541	·901	1·442	1·802	2·795	3·604	9·01	18·02	36·04	90·1	180·2
110°	·1291	·2581	·387	·645	1·032	1·291	1·935	2·581	6·45	12·91	25·81	64·5	129·1
115°	·0972	·1944	·292	·486	·777	·972	1·460	1·944	4·86	0·72	19·44	48·6	97·2
120°	·0750	·1500	·225	·375	·600	·750	1·113	1·500	3·75	7·50	15·00	37·5	75·0
125°	·0585	·1170	·176	·293	·468	·485	·880	1·170	2·93	5·85	11·70	29·3	58·5
130°	·0457	·0913	·137	·228	·365	·457	·685	·913	2·28	4·57	9·13	22·8	45·7
135°	·0354	·0707	·106	·177	·283	·354	·530	·707	1·77	3·54	7·07	17·7	35·4
140°	·0270	·0539	·081	·135	·216	·270	·405	·539	1·35	2·70	5·39	13·5	27·0
145°	·0201	·0402	·060	·100	·161	·201	·300	·402	1·00	2·01	1·02	10·0	20·1
150°	·0144	·0289	·043	·072	·115	·144	·215	·289	·72	1·44	2·89	7·20	14·4
155°	·0099	·0197	·030	·049	·079	·099	·150	·197	·49	·99	1·97	4·90	9·9
160°	·0062	·0125	·019	·031	·050	·062	·095	·125	·31	·62	1·25	3·11	6·2
165°	·0035	·0069	·010	·017	·028	·035	·050	·069	·17	·35	·69	1·72	3·5
170°	·0015	·0031	·005	·008	·012	·015	·025	·031	·03	·15	·31	·80	1·5
175°	·9004	·0008	·001	·002	·003	·004	·005	·008	·02	·04	·08	·21	·4
176°	·0002	·0004	·001	·001	·002	·002	·004	·004	·01	·02	·04	·10	·2
177°	·0001	·0003	·000	·001	·001	·001	·002	·003	·99(5)	·01	·02	·05	·1
178°	·0001	·0001	·000	·000	·000	·001	·001	·001	·00(3)	·00(6)	·01	·03	·0(6)
179°	·0000	·0000	·000	·000	·000	·000	·000	·000	·00	·00	·00	·00	·00
180°	·0000	·0000	·000	·000	·000	·000	·000	·000	·00	·00	·00	·00	·00

3. FINITE DAMPING

We next discuss the case when $\delta \gtrless 0$, and in so doing we have to formulate the expressions for polarisation ratios and refractive indices in such a way that if $\delta \to 0$, these general expressions should reduce to those discussed in the previous section.

In this case F_1 and F_2 are complex roots of the equation (2,13). Let us put

$$F_1=-\rho\, e^{i\phi}, \text{ and consequently } F_2=\frac{1}{\rho}\, e^{-i\phi}, \quad .. \quad (3.1)$$

since $F_1F_2=-1$, with the condition that ρ is always positive. In the particular case $\delta=0$, we have $\phi=0$ or π. Since F_1 is negative for $r<1$, therefore $\phi=0$ for $r<1$. Again F_1 is positive for $r>1$, therefore, $\phi=\pi$ for $r>1$. So we get for $\delta=0$,

$$\rho=\sqrt{1+g^2}-|\,g\,| \text{ for } r\gtrless 1.$$

Now

$$F_1+F_2=\frac{1}{\rho}\, e^{i\phi}-\rho\, e^{-i\phi}=2g\cos\alpha e^{i\alpha}=2G.$$

Equating real and imaginary parts,

$$\left.\begin{aligned}(1-r)\cos\phi\left(\frac{1}{\rho}-\rho\right)-\delta\sin\phi\left(\frac{1}{\rho}+\rho\right)&=-\frac{\omega\sin^2\theta}{\cos\theta}\\ \delta\cos\phi\left(\frac{1}{\rho}-\rho\right)+(1-r)\sin\phi\left(\frac{1}{\rho}+\rho\right)&=0,\end{aligned}\right\} \quad (3.2)$$

or $\cos\phi\{(1/\rho)-\rho\}=2g\cos^2\alpha$,

$\sin\phi\{(1/\rho)+\rho\}=-2g\sin\alpha\cos\alpha$.

Solving the above equations (3.2) we get

$$\left.\begin{aligned}\sin^2\phi&=\frac{2g^2\sin^2\alpha}{1+g^2+\sqrt{1+2g^2\cos 2\alpha+g^4}}\\ \tan\phi&=-\frac{1-\rho^2}{1+\rho^2}\tan\alpha\end{aligned}\right\}. \quad .. \quad (3.3)$$

and $$\rho=\pm\sqrt{1+g^2\cos^2\alpha-\sin^2\phi}\pm\sqrt{g^2\cos^2\alpha-\sin^2\phi}. \quad (3.4)$$

This expression for ρ can reduce to the corresponding relation for $\delta=0$ only if we take

$$\rho=\sqrt{1+g^2\cos^2\alpha-\sin^2\phi}-\sqrt{g^2\cos^2\alpha-\sin^2\phi}. \quad .. \quad (3.4a)$$

Hence $1/\rho-\rho=2\sqrt{g^2\cos^2\alpha-\sin^2\phi}>0$, for all values of α and g. Thus $\rho<1$. Then returning to the equations (3.2), we have for northern hemisphere for the region $r\gtrless 1$, *i.e.*, $g>0$

$\cos\phi\gtrless 0$, $\sin\phi<0$, *i.e.*, $3\pi/2<\phi<2\pi$,

and for the region $r\gtrless 1$, *i.e.*, $g<0$

$\cos\phi\gtrless 0$, $\sin\phi<0$, *i.e.*, $\pi\lesseqgtr\phi<3\pi/2$.

The case in the southern hemisphere is just the opposite. The results can be tabulated as shown in Table II.

With these complex expressions for F_1 and F_2 which reduce to the expressions discussed in the previous chapter, we get the ordinary and extraordinary complex refractive indices as

$$q_0^2=1-\frac{r}{\beta+\omega\cos\theta\, F_1}=1-\frac{r}{\beta-\omega\rho\cos\theta\, e^{i\phi}}=1-\frac{r}{X_0-iY_0} \quad (3.5)$$

TABLE II

θ	r	$g=\frac{\omega \sin^2\theta}{2(r-1)\cos\theta}$	$\delta=0$		$\delta>0$	
			$\rho=-F_1$	ϕ	ρ	ϕ
N. H.	<1	$(+)$	$\sqrt{1+g^2}-g$	0		$3/2\pi<\phi\leq 2\pi$
$\pi/2\leq\theta\leq\pi$	>1	$(-)$	$\sqrt{1+g^2}+\lvert g\rvert$	π	$\sqrt{1+g^2\cos^2\alpha}-\sin^2\phi$	$\pi\leq\phi\leq 3/2\pi$
S. H.	<1	$(-)$	$\sqrt{1+g^2}+\lvert g\rvert$	π	$-\sqrt{g^2\cos^2\alpha}-\sin^2\phi$	$0<\phi<\pi/2$
$0\leq\theta\leq\pi/2$	>1	$(+)$	$\sqrt{1+g^2}-g$	0		$\pi/2<\phi<\pi$

where $X_0=1+\omega\rho\cos\theta\cos\phi$, $Y_0=\delta+\omega\rho\cos\theta\sin\phi$.

$$q_e^2=1-\frac{r}{\beta+\omega\cos\theta F_2}=1-\frac{r}{\beta+\omega/\rho\cos\theta e^{-i\phi}}=1-\frac{r}{X_e-iY_e} \quad .. \quad (3.6)$$

where $X_e=1+\omega/\rho\cos\theta\cos\phi$, $Y_e=\delta+\omega/\rho\cos\theta\sin\phi$.

Following Booker, we may put $q=\mu-(ick/p)$.

Then $$\mu^2-\frac{c^2k^2}{p^2}=1-\frac{rX}{X^2+Y^2},\ \frac{2\mu ck}{p}=\frac{rX}{X^2+Y^2} \quad .. \quad (3.7)$$

and for the non-deviating region, where $ck/p<<1$

$$k=\frac{p}{2c}\left(\frac{1}{\mu}-\mu\right)\frac{Y}{X},\qquad \mu^2=1-\frac{rX}{X^2+Y^2}.$$

We have thus for the non-deviating region:

$$\left.\begin{aligned}\mu_0^2&=1-r\frac{1-\omega\rho\cos\theta\cos\phi}{(1-\omega\rho\cos\theta\cos\phi)^2+(\delta+\rho\omega\cos\theta\sin\phi)^2}\\ k_0&=\frac{p}{2c}\left(\frac{1}{\mu}-\mu\right)\frac{\delta+\rho\omega\cos\theta\sin\phi}{1-\omega\rho\cos\theta\cos\phi}\end{aligned}\right\} \quad (3.8)$$

$$\left.\begin{aligned}\mu_e^2&=1-r\frac{1+\omega/\rho\cos\theta\cos\phi}{(1+\omega/\rho\cos\theta\cos\phi)^2+(\delta+\omega/\rho\cos\theta\sin\phi)^2}\\ k_e&=\frac{p}{2c}\left(\frac{1}{\mu_e}-\mu_e\right)\frac{\delta+\omega/\rho\cos\theta\sin\phi}{(1+\omega/\rho\cos\theta\cos\phi)}\end{aligned}\right\}. \quad (3.9)$$

The correctness of the above expressions can be tested for special cases:

For the magnetic equator, $\theta=\pi/2$ we have from (3.4*a*), $\rho=0$ and $\frac{\sin\theta}{\rho}e^{-i\phi}=\frac{\omega}{\beta-r}$. Hence, we get the equations (2.10), as special cases of (2.16*a*). For the magnetic north pole, $\theta=\pi$, $\rho=1$, $\phi=\pi$, hence (2.16*a*) reduce to equations (2.11). For the magnetic south pole $\theta=0$, $\rho=1$, $\phi=0$, hence (2.16*a*) reduce to equations (2.11*a*).

4. POLARISATION

Let us next discuss the polarisation of the down-coming wave for any station for a stratified, slowly varying ionosphere with finite damping. Since the e.m. waves which are propagated in such a medium are not transverse in the electric vector E, but are transverse in the magnetic vector H and in the method of detection, the H vector is utilized, it is customary to express the polarisation of the waves with respect to the latter. So we start with the equations of propagation of the magnetic vector, *viz.*,

$$\frac{d^2H_x}{du^2}+K_2H_x-tLH_y=0 \quad .. \quad (4.1)$$

$$\frac{d^2H_y}{du^2}+K_1H_y+iLH_x=0 \quad .. \quad (4.2)$$

in place of the corresponding equations (2.6) and (2.8) for the electric vector. Equations (4.1) and (4.2) follow immediately from (2.6) and (2.8) and (2.1). Eliminating H_y and H_x from (4.2) and (4.3) respectively we get

$$\frac{d^4H_x}{du^4}+(K_1+K_2)\frac{d^2H_x}{du^2}+(K_1K_2-L^2)H_x=0 \quad .. \quad (4.3)$$

and $$\frac{d^4H_y}{du^4}+(K_1+K_2)\frac{d^2H_y}{du^2}+(K_1K_2-L^2)H_y=0 \quad .. \quad (4.4)$$

where the derivatives of K_1, K_2 and L have been neglected as before. The general solutions of (4.3) and (4.4) are

$$H_x=A_1e^{is_1u}+A_2e^{+is_2u}+A_3e^{-is_1u}+A_4e^{-is_2u}$$

$$H_y=B_1e^{is_1u}+B_2e^{+is_2u}+B_3e^{-is_1u}+B_4e^{-is_2u}$$

where

$$s_1^2=\frac{K_1+K_2-\sqrt{(K_1-K_2)^2+4L^2}}{2}$$

$$s_2^2=\frac{K_1+K_2+\sqrt{(K_1-K_2)^2+4K^2}}{2}.$$

It can be easily shown that

$$s_1^2 = q_1^2, \qquad s_2^2 = q_2^2.$$

Retaining only the solutions for the downcoming waves, we get

$$H_x = A_1\, e^{iq_2u} + A_2\, e^{iq_1u} \qquad .. \quad (4.5)$$

$$H_y = B_1\, e^{iq_2u} + B_2\, e^{iq_1u} \qquad .. \quad (4.6)$$

where q_1 and q_2 are those roots of s_1^2 and s_2^2 respectively which have the imaginary parts positive. Substituting (4.5) and (4.6) in (4.1) we get,

$$(-q_1^2 A_2 + K_2 A_2 - iLB_2)\, e^{iq_1u} + (-q_2^2 A_1 + K_2 A_1 - iLB_1)\, e^{iq_2u} = 0,$$

which being an identity in u yields

$$-q_1^2 A_2 + K_2 A_2 - iL B_2 = 0$$
$$-q_2^2 A_1 + K_2 A_1 - iL B_1 = 0$$

whence we have, referring back to (4.1) and (4.2),

$$\frac{B_1}{A_1} = i\,\frac{q_2^2 - K_2}{L}\,; \quad \frac{B_2}{A_2} = i\,\frac{q_1^2 - K_2}{L}. \qquad .. \quad (4.7)$$

From the general solutions of (4.1) and (4.2) it is evident that these equations represent two waves given by

$$H_x^{(1)} = A_1\, e^{iq_2u}, \quad H_y^{(1)} = B_1\, e^{iq_1u}$$
$$H_x^{(2)} = A_2\, e^{iq_1u} \quad H_y^{(2)} = B_2\, e^{iq_2u}$$

travelling with complex phase velocities c/q_2 and c/q_1 respectively. Following the nomenclature adopted before $H_x^{(1)}$ & $H_y^{(1)}$ combine to give the downcoming extraordinary wave and $H_x^{(2)}$ & $H_y^{(2)}$ give the downcoming ordinary wave, and the polarisation ratios for the two waves are

$$\frac{H_y^{(0)}}{H_x^{(0)}} = \frac{B_2}{A_2} = i\,\frac{q_1^2 - K_2}{L} = iF_1$$

$$\frac{H_y^{e}}{H_x^{e}} = \frac{B_1}{A_1} = i\,\frac{q_2^2 - K_2}{L} = iF_2.$$

Taking $H_x^{(0)} = R_{ox}\, e^{i(\gamma_{ox}+pt)}$, $H_y^{(0)} = R_{oy}\, e^{i(\gamma_{oy}+pt)}$

where R_{ox}, P_{oy}, γ_{ox} and γ_{oy} are real functions of u, we get

$$H_x^{(0)} = R_{ox} \cos(\gamma_{ox}+pt), \quad H_y^{(0)} = R_{oy} \cos(\gamma_{oy}+pt)$$

as the two true solutions of the problem with

$$E = E_0 \cos pt \text{ in place of } E = E_0\, e^{ipt}.$$

Hence $\dfrac{H_y^{(0)}}{H_x^{(0)}} = \dfrac{R_{oy}}{R_{ox}}\, e^{i(\gamma_{oy}-\gamma_{ox})} = iF_1 = -i\rho e^{i\phi} = \rho e^{i(\phi-\pi/2)},$

whence
$$\rho = \frac{R_{oy}}{R_{ox}}$$

and
$$\gamma_{oy} - \gamma_{ox} = \phi - \frac{\pi}{2}$$

are the ratio of the axes and the constant phase difference between the y and x components of the magnetic vector respectively. The equation of the polarisation ellipse for the ordinary wave follows immediately by eliminating "pt" between the two equations in (4·8): We have

$$H_x^{(o)2} - \frac{2H_x^{(o)} H_y^{(o)}}{\rho} \sin\phi + \frac{H_y^{(o)2}}{\rho} = R_{o\,x}^2 \cos^2\phi. \qquad .. \quad (4·9)$$

This equation shows that the axes of the ellipse are tilted to the respective y and x axes, the amount ψ_o of tilt to the y axis being given by

$$\tan 2\psi_0 = -\frac{2\rho \sin\phi}{1-\rho^2}. \qquad (4.10)$$

The points of contact of this ellipse with the circumscribed rectangle are (Fig. 4) respectively $(\pm R_{ox} \sin\phi, \pm R_{ox}\rho)$ and $(\pm R_{ox} \pm \rho R_{ox} \sin\phi)$

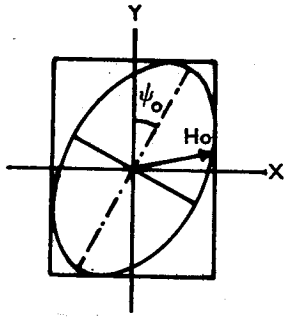

Fig. 4
Polarisation ellipse for the reflected o-wave (northern hemisphere)

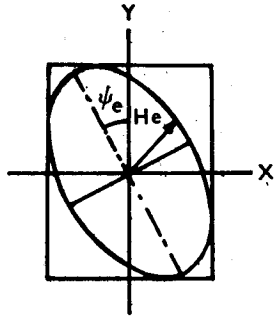

Fig. 5
Polarisation ellipse for the reflected o-wave (southern hemisphere)

For the other wave

$$\rho_e = \frac{R_{ey}}{R_{ex}}, \quad \gamma_{ex} - \gamma_{ex} = \phi + \frac{\pi}{2}$$

and it can easily be shown that

$$\rho_e = \frac{1}{\rho} \text{ and } (\gamma_{ey} - \gamma_{ex}) - (\gamma_{ox} - \gamma_{ox}) = \pi \qquad (4.11)$$

consequently the equation of the polarisation ellipse for the e-wave is

$$\frac{H_x^{e2}}{\rho^2} + \frac{2H_x^e H_y^e}{\rho} \sin\phi + H_y^{e2} = \frac{R_{ex}^2}{\rho^2} \cos^2\phi$$

which shows the same ellipse rotated through an angle $\pi/2$. For this ellipse (Fig. 5) the angle of tilt and the points of contact with the circumscribed rectangle are given by:

$$\tan 2\psi_e = -\frac{2\rho \sin\phi}{1-\rho^2} = \tan(2\psi_o + \pi) \qquad (4.13)$$

and
$$\left(\pm R_e \sin\phi, \pm\frac{R_e}{\rho}\right), \left(\pm R_e, \pm\frac{R_e}{\rho}\sin\phi\right)$$

In the experimental methods of determining the ratio of the axes of the polarisation ellipse, it is generally assumed

that the polarisation of the downcoming wave is mainly determined by the lowest layers of the ionosphere where N the ion concentration tends to vanish. Recently Eckersly (1945) has determined the polarisation of the downcoming waves for $p=6.1$, 6.4 and 7.6 Mc. and has remarked that in order to agree with his experimental results, the polarisation of the downcoming wave should be determined not by the lowest layer of the ionised strata but somewhere inside. Since there is as yet no definite and convincing evidence either experimental or theoretical, of the particular strata or the entire layer fixing the polarisation, we have plotted ρ, the ratio of the axes for the o-wave as a function of $\theta'=\theta-\frac{\pi}{2}$, the magnetic latitude of the place of observations for various values of ω, for r, *i.e.*, N→o.

Sense of rotation of the polarisation ellipse can be inferred from equations (4·8). Since the damping has no effect on the sense of rotation of the magnetic vector, we infer the sense of rotation for the case where damping is absent. In this case for northern hemisphere $\phi=$o, and hence equation (4·8) gives

$$H_x^{\circ}=R_{ox}\cos(\gamma_{ox}+pt)$$
$$H_y^{(\circ)}=\rho R_{ox}\sin(\gamma_{ox}+pt).$$

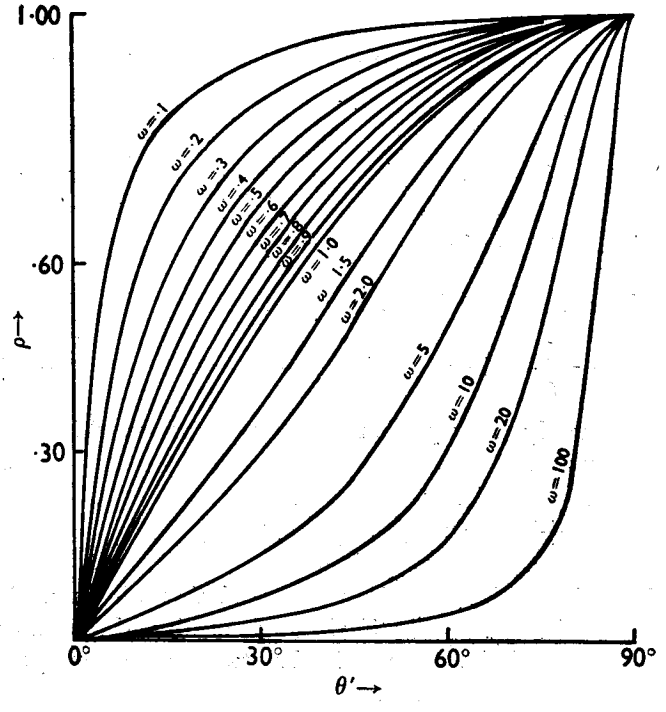

Fig. 6

Variation of the polarisation ratio $=\rho\sqrt{1+g^2}-|g|$ where $g=\frac{\omega\sin^2\theta}{2\cos\theta}$ for various values of $\omega=p_h/p$ for different angles of propagation. $\rho\to1$ means circular polarisation.

Hence as t increases from o, H_x° decreases from $R_{ox}\cos\gamma_{ox}$ to o and remains positive, while $H_y^{(\circ)}$ increases from ρR_{ox} sin γ_{ox} to ρ R_{ox} showing that the vector H_x°, whose components are H_x° and H_y° and which describes the ellipse given by (4·9) is moving in the anticlockwise direction. Thus for all waves received in the northern hemisphere the downcoming ordinary wave is polarised in the anticlock-wise direction as viewed along the direction of propagation. For the extraordinary wave,

$$H_x^e=R_{ex}\cos(\gamma_{ex}+pt)$$
$$H_y^e=-\frac{R_{ex}}{\rho}\sin(\gamma_{ex}+pt).$$

Hence as i increases from o, H_x^e decreases as before but H_y^e becomes more and more negative showing that the vector H^e whose components are H_x^e and H_y^e and which describes the ellipse given by (4·12) moves in the clockwise direction. Thus for all stations in the northern hemisphere, the downcoming e-wave is polarised right handed as viewed along the direction of propagation.

For the southern hemisphere the sense of the rotation of the two ellipses will be just opposite since $\phi=\pi$ for $\delta=$o in the southern hemisphere in place of $\phi=$o for $\delta=$o in the northern hemisphere.

Acknowledgement

Two of us, B. K. Banerjea and U. C. Guha, have been the recipients of two Research scholarships from the Radio Research Committee of the Council of Scientific and Industrial Research (India) and wish to express their thanks to the Council for facility to work and permission to publish this paper.

Palit Laboratory of Physics,
Calcutta University.

References

Appleton 1932, *Jour. Inst. Elec. Eng.*, **71**, 646.
Banerjea 1947, *Proc. Roy. Soc. A.*, **196**, 67.
Booker 1936, *Proc. Roy. Soc. A.*, **155**, 235.
Eckersley 1945. *Proc. Roy. Soc. A.*, **184**, 196.
Darwin 1925, *Trans. Camb. Phil. Soc.*, **23**, 137.
Hartree 1932, *Proc. Camb. Phil. Soc.*, **29**, 143.
Rydbeck 1940, *J. Applied Physics*, **13**, 577.
Saha, Rai and Mathur 1937, *Nat. Inst. Sci. (India)*, **4**, 53.
Saha and Rai 1937, *Nat. Inst. Sci. (India)*, **4**, 319.
Saha and Banerjea 1945, *Ind. Journ. Phy.*, **19**, 159.
Stratton 1942, Electromagnetic Theory, p. 326.