71. THE PROPAGATION AND THE TOTAL REFLECTION OF ELECTROMAGNETIC WAVES IN THE IONOSPHERE

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ABSTRACT

A critical review is given in this article of Prof. S. N. Bose's paper published in this journal. It is shown that when the collision frequency is taken to be zero, his method gives us the same result for the propagation of wireless waves as that of the earlier workers. The conditions of reflection which he has deduced for the case where collision cannot be neglected appear to require revision.

The propagation and the total reflection of electromagnetic waves in the ionosphere has been the subject of numerous investigations within the last ten years, a full bibliography of which is given under the references. A critical review of these papers shows that there are many points connected with this problem which have not yet received adequate explanation. In his pioneering work on the magnetionic theory, Appleton (1932) did not actually solve the relevant Maxwellian Equations but expressions were obtained for the refractive index from a calculation of the dielectric constant of the medium, which is supposed to consist of a number of free electrons and ions. The displacement of these under the magnetic field which is limited by collisions with neutral particles and positive ions constitutes the displacement current, which is necessary to calculate the complex dielectric constant. Appleton's method is usually known as the ray theory of propagation of the electromagnetic waves. The refractive index comes out in general to be a complex quantity and has two different values depending upon the state of polarisation of the wave. Further it is a function of the electron concentration and collisional frequency, both of which are functions of height. Consequently the wave equation becomes too complex for solution. From the analysis, it follows that the original wave splits up into two ordinary and extraordinary, which are propagated with different velocities, as in a doubly refracting medium. He supposes that in the case when collisions can be neglected, the wave gets reflected from the layer where the refractive index becomes equal to zero. This enabled him to obtain the conditions of reflection involving the electron concentration and the frequency of the wave, which are now well-known and have received verification at least in the case of the F-layer.

A number of other methods has been proposed of which we may mention that of Försterling and Lassen (1933), Saha, Rai and Mathur (1938) and that of Hartree (1931) developed further by Booker (1935). The works of these authors lead to the same value of refractive index as that of Appleton, though originally they aimed at obtaining different results.

Recently Prof. S. N. Bose (1938) of Dacca has tackled the same problem by the method of characteristics, used for wave propagation by Hadamard, Debye and others. He confirms in general the conclusions of the previous investigators when collisions can be neglected, but gives new results when the collisions cannot be neglected. His results are, however, expressed in rather unfamiliar symbols, hence it is difficult to compare them with those of earlier workers and apply them to the elucidation of outstanding problems. The object of this paper is to examine his methods and results critically, to express them in a language easily comprehensible to workers on the ionosphere, and to find out how far the results obtained are new.

As we have to make a constant comparison between Bose's paper and the paper previously published by Saha, Rai and Mathur as well as those of other workers, we will refer to the latter as paper I.

The fundamental equations for propagation can be written as

$$\frac{1}{c}\frac{d\vec{E}}{dt} - \text{Curl}\,\vec{H} = -\frac{\rho\vec{V}}{c}. \tag{1}$$

$$\frac{1}{c}\frac{d\vec{H}}{dt} + \text{Curl}\,\vec{E} = 0. \tag{2}$$

$$\text{Div}\,\vec{H} = 0. \tag{3}$$

$$\text{Div}\,\vec{E} = \rho. \tag{4}$$

These equations may be compared with (1.1) of paper I. Bose has used E, while in paper I, $\vec{D}$ was used. But $\vec{D} = K\vec{E} + \vec{P}$, $K=1$, and $\vec{P} = Ne\vec{V}$, where $\vec{V}$ is the velocity of electrons. P is therefore the displacement current, a term denoted by Bose by the symbol θ. For his θ_0, which is electrical density, we have used ρ. The fundamental equations used here, are of the same form as those of Booker.[5] The equations satisfy the conditions of continuity

$$\frac{d\rho}{dt} + \text{Div}\,(\rho\vec{V}) = 0. \qquad (5)$$

Let us now suppose (see Bose, p. 122) that every quantity vary as e^{S}, where S is the phase. For a plane wave

$$S = ip\left[t - \frac{(lx+my+nz)}{c}\right]$$

where p is the pulsatance$=2\pi f$, f being the frequency of the wave, $(l, m, n,)$ direction-cosines of the wave-normal, q=refractive index. In general, (l, m, n) and q may be functions of (x, y, z). Anyhow, no limitation is put on the form of S except that it is a function of (x, y, z, t). Then the above equations reduce to

$$\frac{\dot{S}}{c}\vec{E} - (\vec{\triangle S} \times \vec{H}) = -\frac{\vec{\dot{P}}}{c}. \qquad (1')$$

$$\frac{\dot{S}}{c}\vec{H} + (\vec{\triangle S} \times \vec{E}) = 0. \qquad (2')$$

$$(\vec{\triangle S}.\vec{H}) = 0. \qquad (3')$$

$$(\vec{\triangle S}.\vec{E}) = \rho. \qquad (4')$$

$$\dot{S}\rho + (\vec{P}.\vec{\triangle S}) = 0. \qquad (5')$$

Here the supposition is that if $\vec{E} = \vec{E_0}e^{S}$, $\vec{E_0}$ is a slowly-varying function of (x, y, z, t), *i.e.*,

$$E_0\dot{S} >> \frac{dE_0}{dt},\ E_0\,S_x >> \frac{dE_0}{dx}.$$

From (2), by scalar multiplication with H, we have

$$(\vec{E}.\,\vec{H}) = 0. \qquad (6)$$

From (1), by scalar multiplication with H, we have

$$(\vec{\dot{P}}.\,\vec{H}) = 0. \qquad (7)$$

(4), (6), (7) show that $\vec{E}$, $\vec{\dot{P}}$, $\vec{\triangle S}$ are all normal to $\vec{H}$, and hence lie in the plane perpendicular to $\vec{H}$. But neither E, nor P are in general normal to $\triangle S$, but from (1) we have

$$\frac{I}{c}(\dot{S}\vec{E} + \vec{\dot{P}}) = (\vec{\triangle S} \times \vec{H}). \qquad (8)$$

i.e., the vector $\dot{S}\vec{E} + \vec{\dot{P}}$ is normal to both $\vec{\triangle S}$ and $\vec{H}$.

To find out $\vec{\dot{P}}$, we take the equation of motion of the electrons and ions. Let the displacement of these particles due to the radio-wave be $u = (\xi, \eta, \zeta)$. Then the equation of motion is

$$m\vec{\ddot{u}} = e\vec{E} - g\vec{\dot{u}} + \frac{e}{c}(\vec{\dot{u}} \times \vec{h}). \qquad (9)$$

This is a vector equation, identical with equation (1.3) of our paper I, and equation on p. 131 of Bose. Now we have

$$\vec{P} = N e\vec{u}, \qquad \vec{\dot{P}} = N e\vec{\dot{u}}.$$

Hence replacing $\vec{u}$ by $\vec{P}/Ne$, we have

$$\vec{\ddot{P}} + v\vec{\dot{P}} = \frac{Ne^2}{m}\vec{E} + (\vec{\dot{P}} \times \vec{h}). \qquad (10)$$

Or using the symbols in (1·4) of paper I and putting $\vec{P} = \vec{P_0}e^{S}$

we have $$(\dot{S}+\nu)\vec{\dot{P}} = p_0^2\,\vec{E} + (\vec{\dot{P}} \times \vec{p_h}). \qquad (10')$$

where $$p_h = \frac{eh}{mc},\ p_0^2 = \frac{Ne^2}{m}.$$

This is a vector equation and is equivalent to three different equations.

THE EQUATION OF PROPAGATION

Multiplying (20) by $\triangle S$ vectorially, we have

$$\frac{\dot{S}}{c}(\vec{\triangle S} \times \vec{H}) + \vec{\triangle S} \times (\vec{\triangle S} \times \vec{E}) = 0.$$

since $$(\vec{\triangle S} \times \vec{H}) = \frac{1}{c}(\dot{S}\,\vec{E} + \vec{\dot{P}})$$

$$\vec{\triangle S} \times (\vec{\triangle S} \times \vec{E}) = \vec{\triangle S}\,(\vec{\triangle S}.\,\vec{E}) - E^2(\triangle^2 S)$$

$$= \rho\vec{\triangle S} - \vec{E}\triangle^2 S.$$

Here $\triangle^2 S$ means $(\triangle S)^2$. We have

$$\left(\frac{\dot{S}^2}{c^2} - \triangle^2 S\right)\vec{E} + \frac{\vec{\dot{P}}\,\dot{S}}{c^2} + \rho\vec{\triangle S} = 0. \quad .. \qquad (11)$$

Now making use of the equations (5) and (10), we get the following vector-equation in P

$$\frac{\vec{\dot{P}}\,\dot{S}}{c^2} = \frac{1}{p_0^2}\left[\frac{\dot{S}^2}{c^2} - \triangle^2 S\right]\left[(\dot{S}+\nu)\,\vec{\dot{P}} - (\vec{\dot{P}} \times \vec{p_h})\right] = \frac{\vec{\triangle S}\,(\vec{\dot{P}}.\vec{\triangle S})}{\dot{S}}. \quad .. \quad .. \qquad (12)$$

This is equivalent to three equations, and the operations which we have carried out here is similar to those in § 2 of our paper I.

The three vector equations (12), can be written out in a form more convenient for work by introducing some fresh notation. Here we are closely following Bose's procedure on pp. 137-138 of his paper.

We put

$$\left.\begin{aligned} g(S) &= \frac{\dot{S}^2}{c^2} - \triangle^2 S. \\ p(S) &= \dot{S}(\dot{S}+\nu) + p_0^2 \\ L(S) &= \left(\frac{\dot{S}^2}{c^2} - \triangle^2 S\right)\left[\dot{S}(\dot{S}+\nu) + P_0^2\right] + P_0^2 \triangle^2 S \\ &= g(S)\ p(S) + p_0^2 \triangle^2 S. \end{aligned}\right\} \quad \ldots (13)$$

(12) can now be written as

$$L(S)\vec{\dot{P}} - \dot{S}\ g(S)\ (\vec{\dot{P}} \times \vec{P_h}) = p_0^2 \vec{\triangle S}\ (\vec{\dot{P}}.\ \vec{\triangle S}). \quad (14)$$

Writing out in full, we have

$$\begin{aligned} &\dot{P}_x[L(S) - p_0^2] S_x^2 + \dot{P}_y[T_z - S_x S_y p_0^2] + \dot{P}_z[-T_y - S_x S_y p_0^2] = 0 \\ &\dot{P}_x[-T_z - S_x S_y p_0^2] + \dot{P}_y[L(S) - p_0^2 S_y^2] + \dot{P}_z[T_x - S_z S_y p_0^2] = 0 \\ &\dot{P}_x[T_y - S_x S_z p_0^2] + \dot{P}_y[-T_x - S_x S_z p_0^2] - \dot{P}_z[L(S) - S_z^2 p_0^2] = 0 \end{aligned} \quad \ldots (15)$$

where $\vec{T} = (T_x, T_y, T_z) = -\dot{S}\ g(S)\ \vec{p_h}$.

Since the equations hold simultaneously, the determinant of their co-efficients vanish. From this condition we get after some work

$$L^3(S) - L^2(S) \triangle^2 S p_0^2 + L(S) \dot{S}^2 g^2(S) p_h^2 - \dot{S}^2 g^2(S) p_0^2 (\vec{\triangle \dot{S}} \cdot \vec{p_h})^2 = 0 \quad \ldots (16)$$

Now $\qquad (\triangle S\ .\ p_h) = p_h \triangle S\ .\ \cos \alpha,$

where α is the angle between p_h the direction of the external magnetic field, and $\triangle S$ the wave normal. Further, since

$$L(S) - p_0^2 \triangle^2 S = g(S)\ p(S),$$

we find that $g(S)$ cancels out as a common factor. Equation (16) reduces to

$$p(S)\ L^2(S) - \dot{S}^2\ g(S)\ p_h^2\ L(S) - \dot{S}^2\ g(S)\ p_h^2 p_0^2\ \triangle^2 S \cos^2\alpha = 0 \quad \ldots (17)$$

(16) and (17) are identical with the equations given by Bose on p. 142.

Bose points out that from equation (17), we can calculate the value of the refractive indices. As (17) is a quadratic equation we get two values for the refractive indices, q_1 and q_2. But he does not proceed further to find out the actual values of q_1, and q_2, and compare them with the results of earlier investigators. This we now proceed to do, and we shall show that we get the same value for q_1 and q_2 as obtained on p. 63 of paper I for the o- and x-waves.

Let us put

$$q = \frac{c \mid \triangle S \mid}{\dot{S}} \text{ and } \dot{S} = ip$$

This is equivalent to taking

$$S = ipt \mp \frac{ip}{c} \int q(l dx + m dy + n dz),$$

and for vertical propagation.

$$S = ipt \mp \frac{ip}{c} \int q dz.$$

Now (ql, qm, qn) may be any functions of (x, y, z); q may be called the refractive index. $p = \frac{1}{i}\dot{S}$ is constant, and prescribed by the conditions of the experiment. Then it can be easily verified that if we put

$$1 - q^2 = x, \qquad 1 - \frac{i\nu}{p} = \delta, \qquad r = p_0^2/p^2,$$

we have for vertical propagation, *i.e.*, $S_x = S_y = 0$, $S_z = 1$,

$$L(S) = -\frac{p^4}{c^2}(r - x\beta) \qquad g(S) = -\frac{p^2}{c^2}x \quad p(S) = p^2(r - \beta)$$

and equation (17) reduces to

$$C'x^2 + xr[2\beta(\beta - r) + \omega^2 \sin^2\alpha] - r^2(r - \beta) = 0 \quad \ldots (18)$$

Here C′ is the quantity

$$C' = \beta(\beta^2 - \omega^2) - r(\beta^2 - \omega^2 \cos\alpha)$$

defined in equation (2·18) of paper I.

It can be shown after some work that the roots of equation (18) are given by

$$C'x = C'(1 - q^2) = -r\beta(r - \beta) - \frac{r\omega^2}{2}\sin^2\alpha \left[1 \pm \sqrt{1 + \frac{4(r-\beta)^2 \cos^2\alpha}{\omega^2 \sin^4\alpha}}\right]. \quad (19)$$

We get from (19)

$$C'q^2 = (r - \beta)(\omega^2 - \beta^2 + r\beta) - \frac{r\omega^2}{2}\sin^2\alpha \left[1 \pm \sqrt{\frac{4(r-\beta)^2 \cos^2\alpha}{\omega^2 \sin^4\alpha}}\right] \quad \ldots (20)$$

If we neglect collisions, *i.e.*, put $\beta = 1$, we can easily deduce that (20) reduces to the values of q_o and q_x given in (3·10) of paper I. Hence we have proved that for vertical propagation, Bose's treatment gives the same result as that of earlier workers.

Conditions for Reflection of the E. M. Wave from the Ionosphere

Let us now critically examine Bose's work as far as it deals with the conditions of reflection of the o- and x-waves from the ionosphere. In the original treatment of Appleton, it was supposed that the waves get reflected when q, the refractive index, becomes zero, in the cases where collision frequency can be neglected. This gives the well-known conditions of reflection.

$$\text{o-wave} \qquad \frac{Ne^2}{m}=p^2;$$
$$\text{x-wave} \qquad \frac{Ne^2}{m}=p^2\pm pp_h. \qquad \ldots (21)$$

It is supposed that the reflection represented by + sign does not occur, as the x-wave gets totally reflected from a lower height corresponding to the negative sign. Under these suppositions, we should have for Allahabad for $f=4$ Mc/sec.

$$f_c^o - f_c^x = \cdot 65 \text{ Mc/sec.}$$

While this has been verified in general, Pant and Bajpai (1937), at Allahabad obtained on several occasions, difference of a quite different order:—it was found that for $f=4$ Mc/sec.

$$f_c^o - f_c^x = \cdot 14 \text{ Mc/sec.}$$

This was explained by R. N. Rai (1937) from the idea that waves are returned from the ionosphere, when their group-velocity of propagation becomes zero. From this, he deduced when collisions can be neglected, in addition to the Appleton conditions, a new condition of reflection

$$\frac{Ne^2}{m}=p^2\,\frac{p^2-p_h^2}{p^2-p_L^2}. \qquad \ldots (22)$$

This gives us exactly

$$f_c^o - f_c^x = \cdot 14 \text{ Mc/sec.}$$

Thus the new condition explains completely the result obtained by Pant and Bajpai.

The question now rises: Both Appleton's criterion ($q=0$), as well as that used by Rai (group-velocity=0) are, at best, assumptions. Can they be substantiated as direct deductions from theory ? Further, when the collisional damping cannot be neglected, what will be the condition of reflection ? This is the problem which Bose sets about to solve. His procedure is as follows:

By squaring equation (2), we have

$$\frac{\dot{S}^2}{c^2}H^2=(\triangle S\times E)^2=(\triangle S)^2\,E^2-(\triangle S\,.\,E)^2.$$

$$\text{Hence} \quad \frac{H^2}{E^2}=\frac{c^2\triangle^2 S}{\dot{S}^2}-\frac{(\triangle S.E)^2.c^2}{E^2.\dot{S}^2}=q^2(1-\cos^2\theta), \quad \ldots (23)$$

where θ is the angle between $\triangle$S and E. In general, θ is a definite quantity.

Bose has assumed that reflection takes place when H=0 or when E becomes parallel to $\triangle$S. These conditions reduce to $q=0$ and $q=\infty$ respectively.

From (23) we have

$$H=0, \quad \text{when} \quad q=0,$$
$$\text{and} \quad E=0, \quad \text{when} \quad q=\infty.$$

The propagation loses its wave character either when H=0, or E=0. We can therefore suppose that the wave will be reflected either when $q=0$, or $q=\infty$. The former gives us the conditions of Appleton (equation 21), and the latter gives us Rai's condition.

Bose has further tried to obtain more general conditions of reflection when damping cannot be neglected. (Equations on p. 132, 133 again on p. 139 and 140.). His condition for the o-wave is

$$\frac{dS}{dt}=-\frac{\nu}{2}\pm i\sqrt{p_0^2-\nu^2/4}. \qquad \ldots (24)$$

He concludes that from this the train totally reflected has the form

$$\text{Exp}\left[-\frac{\nu}{2}\,t\pm i\sqrt{p_0^2-\nu^2/4\,t}\,\right],$$

and puts $\qquad p_c^2=p_0^2-\nu^2/4.$

This takes the place of $p^2=p_0^2$ for the o-wave.

Similarly he obtains results for the x-wave on p. 133, and 135, which take the place of

$$p_0^2=p^2\pm pp_h, \qquad p_0^2=p^2\,\frac{p^2-p_h^2}{p^2-p_L^2}.$$

These results obtained by Bose are equivalent to putting the complex values of q for the o and x-waves given by formula (20) equal to zero and infinity. This is shown in Appendix (1), and the work is due to Mr. R. N. Rai.

But it is difficult to agree with this procedure because when q is complex, the conditions of reflection are no longer given by either $q=0$, or $q=\infty$.

For in general, when q is complex we can put $q=\mu+i\dfrac{ck}{p}$.

and it has been shown, that we have for the quasi-transverse as well as quasilongitudinal regions

$$\mu^2=\tfrac{1}{2}\left\{\sqrt{X^2-Y^2}+X\right\},$$
$$\frac{c^2k^2}{p^2}=\tfrac{1}{2}\left\{\sqrt{X^2+Y^2}-X\right\}, \qquad \ldots (25)$$

where X and Y are functions of electron-concentration, and collisional damping. For the o-wave in the equatorial region, we have

$$X=1-\frac{r}{1+\delta^2}, \quad Y=\frac{r\delta}{1+\delta^2}.$$

The forms (25) show that μ^2 and $\dfrac{c^2k^2}{p^2}$ can never be negative. This is at once clear if one looks at the curves drawn for various values of ν, and r by M. Taylor (1938) and Goubau (1935). For the o-wave it is found that for a fixed value of ν, μ gradually decreases with r, and ultimately takes a small value > o, and varying very slowly. For the x-wave, the first part of the curve is similar, but then μ rises abruptly, reaches a steep maximum at the point corresponding to $\mu=\infty$ and then drops out very much as in the μ-curve for the o-wave.

But it should be emphasised that these curves give no idea of the actual variation of μ^2 with z, because μ^2 is a function of N (the number of electrons), and ν, the collision damping, both of which vary continuously with height. For finding out the actual variation of μ^2 with height, we can adopt two procedures—we can take a number of curves of the type drawn by Goubau or M. Taylor for varying values of ν, plot them on cardboards, and then cut the cardboards along the curves. These may then be arranged in a three-dimensional array, behind each other, so that X-axis corresponds to N, the Y-axis to ν, and the Z-axis to μ^2. We have then to take a section through the three dimensional profile of the μ^2-surface, corresponding to the actual conditions in the atmosphere.

The other procedure would be to plot μ^2-values taking some theoretical values for ν and N, the electron concentration. N can be calculated from Chapman's formula for a simple region which has been shown by Saha and Rai (1938) to hold for the general case when radiation need not be monochromatic and ν can be calculated from the kinetic gas theory, by taking T=constant, or T varying according to some assumed law.

In general, $\mu^2=1$ in the non-deviating region, and variations will occur only in the deviating region. The curve will be usually smooth, but may show sudden fluctuations when we pass through irregular banks of ions or electrons, such as may likely be produced by minor causes. When we come to the simple region, μ^2 will vary continuously from unity to a small value, depending on p. These calculations are being carried out by the junior author.

These arguments tell us that both μ^2 and $\left(\frac{c^2k^2}{p^2}\right)$ the real and imaginary parts of q, are essentially positive, and they can be zero, only in the ideal case when δ, the collision frequency is zero. Hence it is not possible to put $q=0$, and deduce any condition from it. In the same way, we cannot put $q=\infty$, when q is complex.

References

Appleton (1932): *J. Elec. Engrs.*, **71**, 642.
Booker (1936): *Proc. Roy. Soc.* A., **155**, 235.
Bose (1938): *Ind. Journ. Phys.*, **21**, 121.
Fösterling & Lassen (1933): *Ann. d. Physik.*, **18**, 26.
Goubau (1935): *Zs. f. H. F. T.*, **45**, 181.
Hartree (1931): *Proc. Camb. Phil. Soc.*, **27**, 143.
Pant & Bajpai (1937): *Science & Culture*, **2**, 409.
Rai (1937): *Proc. Nat. Inst. Sci. India*, **3**, 307.
Saha, Rai & Mathur (1938): *Proc. Nat. Inst. Sci. India*, **4**, 53.
Taylor (1933): *Proc. Roy. Soc.*, **45**, 245.

APPENDIX-I

Equation (20) can be written as

$$C'q_0^2=t(w^2+\beta t)-\frac{rw^2}{2}\sin^2\alpha\left[1-\sqrt{1+\frac{4t^2\cos^2\alpha}{w^2\sin^4\alpha}}\right],\ldots \quad (i)$$

$$C'q_x^2=t\,(w^2+\beta t)-\frac{rw^2}{2}\sin^2\alpha\left[1+\sqrt{1+\frac{4t^2\cos^2\alpha}{w^2\sin^4\alpha}}\right],\ldots \quad (ii)$$

where $t=(r-\beta)$. Now $q_0^2=o$, when $t=o$, or $r-\beta=o$, or $p^2-i\nu p-p_0^2=o$, or $p=\frac{i\nu}{2}\pm\sqrt{p_0^2-\nu^2/4}$.

Therefore the critical frequency is given by

$$p_c=\sqrt{p_0^2-\nu^2/4}. \quad\ldots (iii)$$

$q_x^2=o$ when $t^2=w^2$, or $t=\pm\omega$, or $p^2-i\nu p-p_0^2\pm pp_h=o$.

Putting $p'=p-\frac{i\nu}{2}\pm\frac{p_h}{2}$, we have $p'^2=\frac{p_h^2}{4}-\frac{\nu^2}{4}\mp\frac{i\nu p_h}{2}+p_0^2$
$=(a+ib)^2$,

where

$$a=\left[\tfrac{1}{2}\left\{\frac{p_h^2}{4}-\frac{\nu^2}{4}+p_0^2+\sqrt{\left(\frac{p_h^2}{4}-\frac{\nu^2}{4}+p_0^2\right)^2+\frac{\nu^2p_h^2}{4}}\right\}\right]^{\frac{1}{2}}. \quad (iv)$$

The critical frequency is given by

$$p_c=a\mp p_h/2. \quad\ldots (v)$$

q_0^2 and q_x^2 are equal to infinity when $C'=0$

or
$$\beta(\beta^2-\omega^2)-r(\beta^2-\omega^2\cos^2\alpha)=0.$$

After simplification and rearrangement, we have

$$ip(ip+\nu)+\frac{p_0^2[(ip+\nu)^2+p_h^2\cos^2\alpha]}{[(ip+\nu)^2+p_h^2]}=0,$$

which reduces to

$$p_0^2=p^2\frac{p^2-p_h^2}{p^2-p_h^2\cos^2\alpha}, \quad\ldots (vi)$$

when $\nu=0$.

Here we see that the conditions (*iv*), (*v*), (*vi*) are the same as those obtained by Bose.