theory of the effect of radiation-pressure on the expulsion of the molecules. But the general considerations show *that radiation-pressure may exert an effect on the atoms and molecules which are out of all proportion to their actual sizes. It also shows that the radiation-pressure exerts a sort of sifting action on the molecules, driving the active ones radially outward along the direction of the beam. The cumulative effect of the pulses may be sufficiently great to endow the atoms with a large velocity—the velocity with which the tops of solar prominences are observed to shoot up.*

The velocity of the red prominences are sometimes found to be as high as 6×10^7 cm per second.

The solar prominences have sometimes been explained on the assumption that they are due to the convection of hot masses of vapor from the solar photosphere, which, after reaching the atmosphere, are supposed to expand adiabatically and develop the large velocities with which the prominences are observed to shoot up. But both Pringsheim and Nicholson[10] have pointed out several insuperable difficulties in the way of the acceptance of this hypothesis, including the deduction that the maximum velocity obtainable from adiabatic expansion is less than $\frac{1}{25}$ of the velocity with which the prominences are observed to shoot forward (6×10^7 cm). Nicholson has suggested that some unknown forces of electrical origin may be the cause of these large velocities, but even granting that the electrical fields exist in the sun it is difficult to see how this can act upon the luminous hydrogen particles, which are most probably uncharged. According to the hypothesis put forward in this paper, the effect of radiation-pressure on the separate particles is altogether disproportionate to the dimensions of the particles and may cause them to be endowed with a "levity"[11] long sought for in the explanation of the prominences, the corona, and other solar phenomena, including the extension of the solar atmosphere.[12] The hypothesis presents the problem of the radiative equilibrium of the solar atmosphere in a new light.

These ideas may be applied to the explanation of the tails of comets. The tails of comets are undoubtedly caused by some sort of repulsive action exerted by solar light, but since, on the older theory, the effect was found evanescent on particles of the molecular size, the tail was supposed to consist of some sort of cosmic dust. But the spectroscopic examination of the light from the tails shows that they consist, at least partly, of luminous gases (CO, CO_2)[13]. Now the explanation is quite easy, if the considerations advanced in this paper hold. As the comet approaches the sun, more and more pulses of light from the sun traverse the nucleus and the coma. Light-pulses of suitable frequency are picked up by the gaseous particles, which thus gradually gain in velocity in a direction away from the sun. The cumulative effect of the absorbed pulses may endow the particle with a velocity sufficient for its escape from the main mass of the cometary matter and form into the tail.

It is hoped to develop these ideas further in a future communication.

UNIVERSITY COLLEGE OF SCIENCE, CALCUTTA.
MARCH 4, 1919.

[10] *Monthly Notices*, **74**, 425, 1914.

[11] Ch. Fabry, lecture delivered before the Astronomical Society of France, 1918 (*L'Astronomie*, **32**, 14, 1918).

[12] Attention may be called to a comprehensive paper by D. Brunt (*Monthly Notices*, **73**, 568, 1913), who has shown that neither of the three theories of the equilibrium of the solar atmosphere (isothermal, adiabatic, or radiative) can account for an atmosphere extending to the observed height of the solar atmosphere. The results of the spectroheliographic observations are distinctly unfavorable to Julius' theory of anomalous dispersion (see *Astrophysical Journal*, Papers by Hale, St. John, and others).

[13] Bohr, *loc cit.*

9. ON THE FUNDAMENTAL LAW OF ELECTRICAL ACTION[1]

(*Phil. Mag., Sr. VI*, **37**, 347, 1919)

1.

In the present paper an attempt has been made to determine the law of attraction between two moving electrons, with the aid of the New Electrodynamics as modified by the Principle of Relativity. The problem is a rather old one, and seems to have first occurred in 1835 to Gauss[2], from whom the title of the paper has been borrowed. Before explaining my methods, I shall give a short history of the problem.

About the year 1826 Ampère published his celebrated laws of electrodynamic action, which enable us to calculate, with strict mathematical exactness, the action between two closed electric currents. If we assume that a current of electricity consists of streams of positive and negative charges moving in opposite directions, this action between two closed currents is seen to be composed of the elementary actions between the moving charges, taken two and two. The moving charges, therefore, cannot attract or repel in the same manner as two stationary charges (viz. force $=ee'/r^2$), for in that case the total action would be zero.

[1] Communicated by Prof. D. N. Mallik.

[2] Much of the Introduction is taken from Maxwells' 'Electricity and Magnetism', Chaps. II and XXIII, see especially pp. 483 *et seq.*

The natural assumption is that the law of attraction in this case is quite different, and it depends not only upon mutual distance between the two electrons, but also upon their velocities. This is the problem which Gauss set himself to answer; he does not of course speak of electrons, but of charged particles, which mathematically amounts to the same thing.

Gauss and his followers adopted a deductive method for solving this problem. Ampère had given the law which should subsist between two elements of current, i.e. the currents flowing through an element of length of a circuit in order to account for the action between two closed currents. This law was derived partly from the Geometry of lines, partly from experiments, and besides, involved a number of assumptions. The solution was therefore not quite convincing, and, indeed, as Grassmann[3] and Stefan[4] subsequently proved, was not a unique one. Three other expressions were found to be as good as Ampère's expression for the action between two elements of current. Still, Ampère's solution seemed to be most likely, because the assumptions were simpler in this than in other cases.

Starting with Ampère's expression for the action between two elements of current, and introducing the further assumption that the current consists of discrete charged particles in motion, Gauss deduced the following expression for the mutual attraction between two charges:

$$F=\frac{ee'}{r^2}\left[1+\frac{1}{c^2}\left(u^2-\frac{3}{2}\left(\frac{du}{dt}\right)^2\right)\right],$$

where e, e' are the charges, r=mutual distance, u=relative velocity.

But the law was found to be inconsistent with the principle of conservation of energy, and naturally fell through.

Other physicists in turn took up the problem. The most celebrated formula is that of Weber[5], according to whom the mutual potential of two moving charges is given by the expression

$$\psi=\frac{ee'}{r}\left[1-\frac{1}{2c^2}\left(\frac{\partial r}{\partial t}\right)^2\right].$$

This formula is consistent with the principle of conservation of energy, but was nevertheless found by Helmholtz[6] to lead to improbable results.

These laws were all based on the idea of action at a distance. But in 1845, Gauss[7] again returned to the problem (which he now calls the real keystone of electrodynamics), with the idea that the action, instead of being propagated instantaneously, may be propagated with a finite velocity in a manner similar to that of light. But he did not succeed, as he himself tells us, in forming any consistent mental picture of the manner in which the action is propagated, and seems to have given up the attempt.

Three other mathematicians, Riemann, Neumann and Betti[8] followed in the wake of Gauss, and suggested solutions, but these also have been no more successful than their predecessors. According to Riemann[9], the force components between two charges are given by the Lagrangian derivatives of the function

$$\psi=\frac{ee'}{R}\left[1-\frac{(u-u')^2+(v-v')^2+(w-w')^2}{c^2}\right],$$

where (u, v, w) are the velocities of the one particle, (u', v', w') are the velocities of the other.

According to all of these theories, the action depends on the relative velocity of the two particles. This can be at once perceived by a reference to the formulae of Gauss, Weber and Riemann. If both particles move with the same velocity, the action would be the same as that between two stationary ones, and there would not be any electrodynamical action. This is a very objectionable feature of these theories, and attention to this fact was first drawn, I believe, by Clausius.[10] Clausius is also the author of a series of elaborate investigations on this point. According to his theory, the components of the force between two electrified particles are the Lagrangian derivatives of the function

$$\phi=\frac{ee'}{R}\left[1-\frac{uu'\cos\theta}{c^2}\right],$$

u and u' being the velocities of the two particles, θ being the angle between their directions of motion. The force components are given by the expressions

$$X=\frac{\partial\phi}{\partial x}-\frac{d}{dt}\left(\frac{\partial\phi}{\partial\frac{dx}{dt}}\right),\qquad Y=\frac{\partial\phi}{\partial y}-\frac{d}{dt}\left(\frac{\partial\phi}{\partial\frac{dy}{dt}}\right),$$

$$Z=\frac{\partial\phi}{\partial z}-\frac{d}{dt}\left(\frac{\partial\phi}{\partial\frac{dz}{dt}}\right).$$

It will be observed that the action depends not upon the relative velocity, but upon the absolute velocities of the two particles. Clausius indeed proceeds to show that his formula, besides leading to Ampère's laws of Electrodynamic action, is remarkably free from the objections which were raised against the other formulae.

Clausius's formula may be said, in a way, to have been confirmed by the investigations of J. J. Thomson.[11] Thomson investigated, from Maxwell's theory of moving tubes of force, the action between two spheres of radii a and a',

[3] Loc. cit. p. 174.
[4] *Populäre Schriften-Boltzmann*, pp. 95 & 96.
[5] Maxwell, loc. cit. pp. 484 & 485.
[6] *Phil. Mag.*, December 1872.
[7] Maxwell, loc. cit. p. 490.
[8] Maxwell, loc. cit. p. 490.
[9] Clausius, *Phil. Mag.*, 1880.
[10] *Journal für Mathematic* (Crelle's Journal), Vols. lxxxii & lxxxiii; *Phil. Mag.*, 1880.
[11] *Phil. Mag.*, 1881. 'Application of Dynamics to Problems of Physics and Chemistry', Chap IV.

moving with the velocities u and u' and carrying the charges e and e'. The kinetic energy was found to be

$$\left(\tfrac{1}{2}m+\tfrac{2}{15}\frac{\mu e^2}{a}\right)u^2+\left(\tfrac{1}{2}m'+\tfrac{2}{15}\frac{\mu e'^2}{a'}\right)u'^2 +\frac{\mu ee'\cos\theta\, uu'}{3R}$$

Neglecting the terms due to the Mass-motion, the Lagrangian Function T-U, for the two charged particles, is easily seen to be equivalent to

$$\frac{\mu}{3}\left(\frac{uu'\cos\theta}{R}\right)-\frac{1}{kR}.$$

The similarity of this form with the Clausius form is apparent. There is of course discrepancy in the $\left(\frac{\mu}{3}\right)$ term.

These formulae are all limited to the case where the velocities of the moving charges are small compared with the velocity of light.

From what has been said before, it will be seen that the problem is still an open one. The investigations hitherto given are largely empirical, and not based on sufficient theoretical basis. In view of the recent extraordinary development of electronic physics, it cannot be said that the importance of the problem has been in any way diminished. On the contrary, a knowledge of the laws of electronic attraction and a clear formulation of the dynamics of the electron are necessary before we can satisfactorily handle any problem on electronic physics,—such as the atomic model, or radiation from atoms and electrons.

2.

In the present investigation I have throughout used the New Electrodynamics (i.e., as modified by Lorentz, Einstein, and Minkowski according to the Principle of Relativity). I have particularly used the method of four-dimensional analysis which was first initiated by Minkowski.[12] A large amount of work in this line has been done by Born[13] and Sommerfeld,[14] though not always with the same specific purpose which I have in this paper. Sommerfeld in particular, in connexion with his development of four-dimensional analysis, has investigated the law of attraction between two moving electrons; but the result obtained is so cumbrous as to make further progress almost impossible. This is due to the fact that for the scalar and vector potential of the field produced by a moving electron, they arrived at an expression which is only a partial statement of the complete result (see remarks at the end of § 8). When this complete result is introduced, the electric and magnetic forces as well as the ponderomotive force acting on an electron come out in very elegant forms, enabling us ultimately to write out the equations of motion of two electrons round each other in a Lagrangian form. When one electron is at rest, the equations lead to Darwin's results (*Phil. Mag.*, 1915).

[12] H. Minkowski, *Mathematische Annalen*, vol. lxviii. p. 472 *et seq.*

[13] Born, *Ann. d. Physik*, vol. xxviii, p. 571.

[14] Sommerfeld, *Ann. d. Physik*, vol. xxxiii. pp. 649 *et seq*; vol. xxxii pp 749 *et seq.*

3. *Notation*

The notation used in this paper is identical with that used by Minkowski and Sommerfeld in the memoirs just mentioned, and is to be found in any one of general treatises on Relativity (Cunningham or Silberstein). However, for the convenience of the reader, it is explained below.

The unit of time used in this paper is $\frac{1}{c}$ times the ordinary unit (c, velocity of light measured in ordinary C. G. S. units), so that with this notation, the velocity of light becomes unity.

We shall, in most cases, use $l=\sqrt{-1}\,t$ so that (x, y, z, l) denotes the space-time coordinates of world-point (Weltpunkt). The quantities

$$(\omega_1, \omega_2, \omega_3, \omega_4)=\frac{1}{\sqrt{1-u^2}}(u_1, u_2, u_3, \sqrt{-1}),$$

where (u_1, u_2, u_3) are the ordinary space components of the velocity of a material point, will denote the space-time components of the Velocity-four-vector. It should be noticed that $(u_1, u_2, u_3)=\frac{d}{dt}(x, y, z)$, and if by τ we denote the proper-time (Eigenzeit) of motion of the material point, we shall have $d\tau=dt\sqrt{(1-u^2)}$, and

$$(\omega_1, \omega_2, \omega_3, \omega_4)=\frac{d}{d\tau}(x, y, z, l).$$

a will denote a four-vector of which the space components are equivalent to the vector-potentials used in Electrodynamics, the time-component $=\sqrt{-1}\,\phi$, where ϕ is the ordinary scalar-potential. This is known as the Potential-four-vector.

The operator

$$\left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}+l\frac{\partial}{\partial l}\right),$$

which plays the same role in the four-dimensional analysis as the familiar operator ∇ in three-dimensions

$$\left(\nabla=i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right),$$

was called by Minkowski "Lor", in honour of H. A. Lorentz, the discoverer of the Principle of Relativity.

It is denoted by $\Box$.

The operator

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}+\frac{\partial^2}{\partial l^2}\right),$$

which corresponds to the three-dimensional operator

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right),$$

is generally denoted by $\Box^2$.

The set of four quantities $\rho(u_1, u_2, u_3, i)$, where $\rho=$density of electricity at a point, is a four-vector according to Lorentz and Einstein. It is known as the Stream-four-vector and will be denoted by S.

4.

The potential-four-vector **a** satisfies the equations[15]

$$\Box^2\mathbf{a}=-4\pi S, \text{ or } \Box^2\mathbf{a}=0, \qquad \ldots(1)$$

according as the world-point at which $\Box^2\mathbf{a}$ is taken is occupied by a stream-four-vector or is empty.

a satisfies also the equation

$$\text{Div } \mathbf{a}, \text{ or } (\Box\mathbf{a})=0 \qquad \ldots(2)$$

at all points of the world-space.

Now the fundamental solution of equations (1), due to a single stream-four-vector S, occupying the world-point (x', y', z', l') is

$$\frac{AS}{r^2}, \text{ or } A\frac{S}{(x-x')^2+(y-y')^2+(z-z')^2+(l-l')^2}, \qquad \ldots(3)$$

where (x, y, z, l) is the world-point at which **a** is to be estimated.

A can be proved to be equivalent to $\frac{1}{\pi}$.

Therefore the potential-four-vector at a world-point (x, y, z, l) due to a distribution in the world-space of the stream-four-vector S is

$$\mathbf{a}=\frac{1}{\pi}\iiiint\frac{S dx' dy' dz' dl'}{(x-x')^2+(y-y')^2+(z-z')^2+(l-l')^2}, \quad (3')$$

N.B. In modern methods of treating problems of Electrodynamics, the usual practice is to choose a unit of current which is $\sqrt{4\pi}$ times smaller than the ordinary unit, thereby instead of having $\Box^2\mathbf{a}=-4\pi S$, we have $\Box^2\mathbf{a}=-S$. I have struck to the older method, because this is more convenient for our purpose.

The fundamental solution $\frac{1}{r^2}$ seems to have been first obtained by Poincaré.[16] It corresponds to the solution $\frac{1}{r}$ in three-dimensional problems on Potential, and is a particular case of the following general result first obtained by Poincare.[17]

If $(x_1, x_2, \ldots x_n)$ be the coordinates of a point in space of n—dimensions, the fundamental solution of the generalized Laplacian

$$\left(\frac{\partial^2}{\partial x_1{}^2}+\frac{\partial^2}{\partial x_2{}^2}+\ldots\ldots\frac{\partial^2}{\partial x_n{}^2}\right) V=0,$$

is $\frac{A}{r^{n-2}}$, where $r^2=(x_1-x_1')^2+(x_2-x_2')^2+\ldots(x_n-x_n')^2$ $\ldots(4)$

5. *Potential-four-vector at an external point due to the motion of a point-charge.*

By a point-charge is meant a charge having no extension in ordinary space. In four-dimensions, however, it has extension in one direction, viz, in the direction of the time-axis if the electron be stationary, or along an axis making an angle of $(\tan^{-1}u)$ with the time-axis, if u be its velocity of motion.

Let (x, y, z, l) be the coordinates of the point-charge, which we suppose to have started from the origin at time $t=0$. Then we have $(x, y, z)=-\sqrt{-1}\ (u_1, u_2, u_3)$. Let (a, b, c, λ) be the coordinates of the external point at which the potential **a** is sought. According to the general theorem in the previous section, the potential-four-vector **a** is given by the integral

$$\int_{-\infty}^{\infty}\frac{\rho_o(\omega)dl'}{(x-a)^2+(y-b)^2+(z-c)^2+(l-\lambda)^2},$$

where $\rho_o(\omega)=\rho\ (u_1, u_2, u_3), \sqrt{-1}),$

and therefore $\rho_o=\rho\sqrt{(1-u^2)}$, the rest-density, which is an invariant according to Lorentz and Einstein,

dl'=an element of length along the axis of motion; dl' is easily seen to be equivalent to $dl\sqrt{(1-u^2)}$.

Now $(x-a)^2+(y-b)^2+(z-c)^2+(l-\lambda)^2$

$$=l^2\ (1-u^2)+2il\ (u_1a+u_2b+u_3c+i\lambda)+a^2+b^2+c^2+\lambda^2$$
$$=l'^2+2il'\ (\omega_1a+\omega_2b+\omega_3c+\omega_4\lambda)+a^2+b^2+c^2+\lambda^2.$$

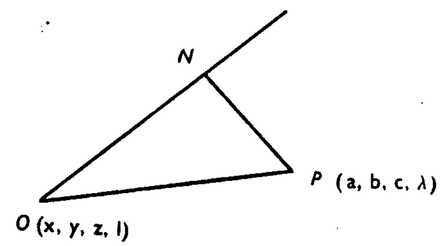
Putting $l'=l\sqrt{(1-u^2)}$,

this integral is easily seen to be equivalent to

$$\frac{\rho_o(\omega)}{[a^2+b^2+c^2+\lambda^2+(a\omega_1+b\omega_2+c\omega_3+\lambda\omega_4)^2]^{\frac{1}{2}}} \qquad \ldots(5)$$

With the aid of four-dimensional geometry, we can give an interesting interpretation to this expression. The direction of motion of the charge (ρ) is given by the line

$$\frac{x}{\omega_1}=\frac{y}{\omega_2}=\frac{z}{\omega_3}=\frac{l}{w_4}.$$



Let P be the point (a, b, c, λ). Then we have

$$PN^2=OP^2-ON^2$$
$$=(a^2+b^2+c^2+\lambda^2)+(a\omega_1+b\omega_2+c\omega_3+\lambda\omega_4)^2,$$

for ON=projection of OP on $OA=i(\omega_1a+\omega_2b+\omega_3c+\omega_4\lambda)$.

Thus the denominator in the expression (5) is seen to be equivalent to R, where R is the perpendicular distance from the external point on the axis of motion.

The result can also be easily proved if we introduce a Lorentz-transformation, by which the axis of motion becomes the new-time-axis. Then in the expression (4), the four-vector $\rho_o(\omega)$ becomes

$$\rho_o\ (0, 0, 0, \sqrt{-1}),$$

and the problem is reduced to one at rest. The denominator becomes equivalent to $R^2+l'^2$ where R is the perpendicular from P on the axis of motion.

[15] Born, *Ann. d. Physik*, vol xxviii. p. 571.
[16] Sommerfeld, *Ann. D. Physik*, vol. xxxiii. p. 663.
[17] *Thèorie du Potential Newtonien.*

We have therefore

$$\mathbf{a}'=\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{\rho_o(i)dl'}{R^2+l'^2}=\frac{\rho_o(0,0,0,i)}{R},$$

Now $(0, 0, 0, \mathbf{a}_4)$ are the components, in the transformed system, of potential-four-vector $\mathbf{a}'$ whose components in the original system are $(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4)$. Re-transforming to the original system, we have

$$[\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4]=\frac{\rho_o(\omega_1, \omega_2, \omega_3, \omega_4)}{R} \quad \ldots .(6)$$

Otherwise—When by means of an orthogonal Lorentz-transformation, we transform from the system (x, y, z, l) to the system (x', y', z', l'), the generalised Laplacian $\Box^2\mathbf{a}$ is transformed to

$$\left(\frac{\partial^2}{\partial x'^2}+\frac{\partial^2}{\partial y'^2}+\frac{\partial^2}{\partial z'^2}+\frac{\partial^2}{\partial l'^2}\right)\mathbf{a}=0, \textit{ or } -4\pi S'.$$

In the present case, the distribution on an infinite line is along the l' axis. Therefore $\mathbf{a}$ must be independent of l' from which

$$\mathbf{a}=\frac{S'}{\sqrt{x'^2+y'^2+z'^2}}.$$

$\sqrt{x'^2+y'^2+z'^2}$ is easily seen to be equivalent to what we have called R previously.

Thus according to this method of investigation also, the potential-four-vector

$$\mathbf{a}=\frac{\rho_o\,(\omega_1, \omega_2, \omega_3, \omega_4)}{R}, \quad \ldots ..(6)$$

where R=perpendicular distance from the external point (a, b, c, λ) on the axis of motion of the point-charge:—direction cosines $\sqrt{-1}\ (\omega_1, \omega_2, \omega_3, \omega_4)$.

6. *The Ponderomotive Force*[18].

If $\mathbf{a}$ be the potential-four-vector in an electric field, and ρ be the electric space-density at a point, the force acting on this point is given by the matrix

$$\rho\begin{vmatrix} u_1, & u_2, & u_3, & i \\ \frac{\partial}{\partial x}, & \frac{\partial}{\partial y}, & \frac{\partial}{\partial z}, & \frac{\partial}{\partial l} \\ \mathbf{a}_1, & \mathbf{a}_2, & \mathbf{a}_3, & \mathbf{a}_4 \end{vmatrix} \quad \ldots\ldots(7)$$

It should be noticed that the word "Force" is used in a generalised sense. The components of this four-vector are (X, Y, Z) the ordinary space-components, and

$$L=i(Xu_1+Yu_2+Zu_3),$$

i.e., $\sqrt{-1}$ times the rate of doing work. The four components are connected by the equation

$$X\omega_1+Y\omega_2+Z\omega_3+L\omega_4=0,$$

i.e., the force-four-vector is always normal to the velocity-four-vector.

Writing (ϕ, F, G, H) for $(\sqrt{-1}\ \mathbf{a}_4, \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$ and introducing the ordinary C.G.S. units, it can be easily verified that this expression is identical with Lorentz's expression for Ponderomotive force.

We shall now write $(\omega_1, \omega_2, \omega_3, \omega_4)$ instead of $(u_1, u_2, u_3, \sqrt{-1})$. Then

$$[X, Y, Z, L]=\rho_o\begin{vmatrix} \omega_1 & \omega_2 & \omega_3 & \omega_4 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} & \frac{\partial}{\partial l} \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 \end{vmatrix}.$$

$\rho_o=\rho\sqrt{(1-u^2)}$ is an invariant, and is generally known as the rest-density,

and $$(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4)=\frac{\rho_o'\omega'}{R'}.$$

R'=perpendicular distance from the external point (x, y, z, l) on the axis of motion of the charge ρ' which produces the field.

The coordinates here refer to the coordinates of the point-charges.[19]

7. *Law of Attraction between two Point-charges.*

We have now

$$X=\rho_o\left[\left\{\omega_1\frac{\partial\mathbf{a}_1}{\partial x}+\omega_2\frac{\partial\mathbf{a}_2}{\partial x}+\omega_3\frac{\partial\mathbf{a}_3}{\partial x}+\omega_4\frac{\partial\mathbf{a}_4}{\partial x}\right\} -\left\{\omega_1\frac{\partial}{\partial x}+\omega_2\frac{\partial}{\partial y}+\omega_3\frac{\partial}{\partial z}+\omega_4\frac{\partial}{\partial l}\right\}\mathbf{a}_1\right],$$

i.e.,

$$X=\rho_o\rho_o'\left[(\omega_1\omega_1'+\omega_2\omega_2'+\omega_3\omega_3'+\omega_4\omega_4')\frac{\partial}{\partial x}\left(\frac{1}{R'}\right) -\left(\omega_1\frac{\partial}{\partial x}+\omega_2\frac{\partial}{\partial y}+\omega_3\frac{\partial}{\partial z}-\omega_4\frac{\partial}{\partial l}\right)\left(\frac{\omega_1'}{R'}\right)\right]$$

for in the expression for $\mathbf{a}$, R is the only term explicitly involving the coordinates (x, y, z, l), (ρ_o', w') being independent of them.

Now let $d\tau$=proper time (Eigenzeit) of motion of A. Then $d\tau=dt\sqrt{(1-u^2)}$,

and $(\omega_1, \omega_2, \omega_3, \omega_4)=\frac{d}{d\tau}(x, y, z, l)$.

We have therefore

$$\frac{d}{d\tau}=\frac{\partial}{\partial x}\cdot\frac{dx}{d\tau}+\frac{\partial}{\partial y}\cdot\frac{dy}{d\tau}+\frac{\partial}{\partial z}\cdot\frac{\partial z}{\partial\tau}+\frac{\partial}{\partial l}\cdot\frac{dl}{d\tau}.$$
$$=\omega_1\frac{\partial}{\partial x}+\omega_2\frac{\partial}{\partial y}+\omega_3\frac{d}{dz}+\omega_4\frac{\partial}{\partial l}$$

If we now put

$$\phi=\frac{\omega_1\omega_1'+\omega_2\omega_2'+\omega_3\omega_3'+\omega_4\omega_4'}{R'}\rho_o\rho_o', \quad \ldots ..(8)$$

then, since

$$(\rho_o\rho_o'\omega_1'/R')=\frac{\partial\Phi}{\partial\omega_1},$$

[18] Minkowski, loc. cit. § 11.

[19] The matrix used for expressing the Ponderomotive Force (X, Y, Z, L) has not been used in the conventional sense (Sommerfeld, *Ann. der Physik*, Vol. xxxii & xxxiii) as can be easily observed.

we have

$$X=\frac{\partial\Phi}{\partial x}-\frac{d}{d\tau}\left(\frac{\partial\Phi}{\partial\omega_1}\right).$$

Similarly for the other components (Y, Z, L).

The form is Lagrangian, and the expression for (X, Y, Z, L) comes out in the form originally pointed out by Clausius.

We therefore prove that the force-four-vector on (A) can be put into the Lagrangian forms

$$\left.\begin{aligned} X&=\frac{\partial\Phi}{\partial x}-\frac{d}{d\tau}\left(\frac{\partial\Phi}{\partial\omega_1}\right)\\ Y&=\frac{\partial\Phi}{\partial y}-\frac{d}{d\tau}\left(\frac{\partial\Phi}{\partial\omega_2}\right)\\ Z&=\frac{\partial\Phi}{\partial z}-\frac{d}{d\tau}\left(\frac{\partial\Phi}{\partial\omega_3}\right)\\ L&=\frac{\partial\Phi}{\partial l}-\frac{d}{d\tau}\left(\frac{\partial\Phi}{\partial\omega_4}\right)\end{aligned}\right\}\quad\ldots\ldots(9)$$

Similarly, if R=perpendicular distance of the point B (a, b, c, λ) from the axis of motion of (A) (x, y, z, l), i.e.,

$$R^2=(x-a)^2+(y-b)^2+(z-c)^2+(l-\lambda)^2$$
$$+[(x-a)w_1+(y-b)w_2+(z-c)w_3+(l-\lambda)\,w_4]^2,$$

and Φ' denotes the expression

$$\frac{\rho\rho_0'}{R'}\,(w_1w_1'+w_2w_2'+w_3w_3'+w_4w_4'),$$

the forces exerted by A on B are given by the equations

$$\left.\begin{aligned} X'&=\frac{\partial\Phi'}{\partial a}-\frac{d}{d\tau'}\left(\frac{\partial\Phi'}{\partial\omega_1'}\right) & Y'&=\frac{\partial\Phi'}{\partial b}-\frac{d}{d\tau'}\left(\frac{\partial\Phi'}{\partial\omega_2'}\right)\\ Z'&=\frac{\partial\Phi'}{\partial c}-\frac{d}{d\tau'}\left(\frac{\partial\Phi'}{\partial\omega_3'}\right) & L'&=\frac{\partial\Phi'}{\partial\lambda}-\frac{d}{d\tau'}\left(\frac{\partial\Phi}{\partial\omega_4'}\right)\end{aligned}\right\}\quad(10)$$

8. *Two Electrons in Motion.*

In the foregoing sections we treated the case of two point-charges. We shall now take the case of two electrons when these are in a state of motion. It will be shown that the same equations would hold if instead of the rest densities ρ_0, ρ_0', we substitute the invariant charges (e, e') and suppose the whole charge to be concentrated at the centre of each.

The electron occupies the space

$$(x-x_0)^2+(y-y_0)^2+(z-z_0)^2 \lesseqgtr r^2;$$

where (x, y, z) are the space-components of any point within the electron, (x_0, y_0, z_0) the corresponding quantities for the centre, and r is the radius.

In three-dimensions this equation represents a sphere, but in four-dimensions this represents a spherical cylinder having infinite extension along the time-axis. The equation shows that the electron is at rest.

We shall now write down the equation of a spherical electron moving with a uniform velocity (u_1, u_2, u_3).

In three-dimensions, the equation of a circular cylinder having the line

$$\frac{x-x_0}{l}=\frac{y-y_0}{m}=\frac{z-z_0}{n}$$

as the axis is given by the equation

$$(x-x_0)^2+(y-y_0)^2+(z-z_0)^2-[l(x-x_0)+m(y-y_0)+n(z-z_0)]^2=r^2.$$

Similarly, in four-dimensions, since the axis of motion is given by

$$\frac{x-x_0}{i\omega_1}=\frac{y-y_0}{i\omega_2}=\frac{z-z_0}{i\omega_3}=\frac{l-l_0}{i\omega_4},$$

therefore the equation of the cylinder having this as axis is

$$(x-x_0)^2+(y-y_0)^2+(z-z_0)^2+(l-l_0)^2+[w_1(x-x_0)+w_2(y-y_0)+w_3(z-z_0)+w_4(l-l_0)]^2=r^2.$$

That this is so can easily be observed by introducing a Lorentz-transformation in which the line of motion is the new time-axis, and the velocity is equivalent to the moment of transformation. Then if (ξ, η, ζ, ν) be the new coordinates, we have

$$(\xi-\xi_0)^2+(\eta-\eta_0)^2+(\zeta-\zeta_0)^2+(\nu-\nu_0)^2$$
$$=(x-x_0)^2+(y-y_0)^2+(z-z_0)^2+(l-l_0)^2$$

and

$$i[\omega_1(x-x_0)+\omega_2(y-y_0)+\omega_3\,(z-z_0)+\omega_4(l-l_0)]=\nu-\nu_0.$$

$\therefore$ the equation of the electron becomes

$$(\xi-\xi_0)^2+(\eta-\eta_0)^2+(\zeta-\zeta_0)^2=r^2.$$

We shall now calculate the potential-four-vector due to the motion of electron at an external point (a, b, c, λ). We have

$$\mathbf{a}=\frac{1}{\pi}\iiiint\frac{\rho_0(\omega)\,dxdydzdl}{[(x-a)^2+(y-b)^2+(z-c)^2+(l-\lambda)^2]},\quad(10a)$$

the integration being extended over the whole world-space enclosed by the electron.

We shall now introduce again the above-mentioned Lorentz-transformation. Then we can write

$d\xi d\eta d\zeta d\nu$ for $dx\,dy\,dz\,dl$,

$\rho_0(0, 0, 0, i)$ for $\rho_0(\omega_1, \omega_2, \omega_3, \omega_4)$,

and $(\xi-\xi')^2+(\eta-\eta')^2+(\zeta-\zeta')^2+(\nu-\nu')^2$ for $(x-a)^2+(y-b)^2+(z-c)^2+(l-\lambda)^2$.

Now $\mathbf{a}'$ the transformed of $\mathbf{a}$ becomes

$$=\frac{1}{\pi}\iiiint\frac{\rho_0(0,0,0,i)\;d\xi d\eta d\zeta d\nu}{(\xi-\xi')^2+(\eta-\eta')^2+(\zeta-\zeta')^2+(\nu-\nu')^2},$$

integrated over the world-space

$$(\xi-\xi_0)^2+(\eta-\eta_0)^2+(\zeta-\zeta_0)^2 \lesseqgtr r^2.\quad(A).$$

We shall first integrate over the new time-axis. The limits are then from $-\infty$ to $+\infty$.

$$\therefore\quad \mathbf{a}'=\iiint\frac{\rho_0(0,0,0,i)\;d\xi d\eta d\zeta}{\sqrt{(\xi-\xi')^2+(\eta-\eta')^2+(\zeta-\zeta')^2}},$$

over the spherical volume (A). This is a three-dimensional potential problem, and is easily seen to be

$$=\frac{e(0,0,0,i)}{\sqrt{(\xi_0-\xi')^2+(\eta_0-\eta')^2+(\zeta_0-\zeta')^2}},$$

where $e=\iiint \rho_0 d\xi d\eta d\zeta$,

integrated over the spherical volume (A).

Now $\sqrt{(\xi_0-\xi')^2+(\eta_0-\eta')^2+(\zeta_0-\zeta')^2}$ is the perpendicular distance from the external point $(\xi', \eta', \zeta' \nu')$ on the axis of motion; we can denote this by R.

Then

$$\mathbf{a}'=\frac{(0, 0, 0, i)e}{R}.$$

Now $\mathbf{a}'$ is what the potential-four-vector $\mathbf{a}$ with the components $(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4)$ becomes when the transformation is introduced. Retransforming to the original coordinates, we have

$$[\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4]=\frac{(w_1, w_2, w_3, w_4)e}{R} \qquad \ldots(11)$$

We can now express R in terms of the original system of coordinates.

$$R^2=(x_0-a)^2+(y_0-b)^2+(z_0-c)^2+(l_0-\lambda)^2+$$
$$[(x_0-a)w_1+(y_0-b)w_2+(z_0-c)w_3+(l_0-\lambda)w_4]^2,$$

where (x_0, y_0, z_0, l_0) are the coordinates of the centre of the electron, (a, b, c, λ) those of the external point.

N.B. The Scalar and Vector potentials due to the motion of an electron were first obtained by Lienard and Wiechert[20] about 1898. They were expressed in the form

$$\phi=\frac{e}{r\left(1-{}^{u_r}/_c\right)}, \quad [\mathrm{F, G, H}]=\frac{e\left(\frac{u_1}{c}, \frac{u_2}{c}, \frac{u_3}{c}\right)}{r\left(1-{}^{u_r}/_c\right)}, \ldots(12)$$

where r is the distance of the external point from the point occupied by the electron at a time $\left(t-{}^{u_r}/_c\right)$, etc. (u_1, u_2, u_3) are the velocity components at the time $(t-r/c)$, u_r is the component of this velocity along the line of r.

The expression (11) is in fact equivalent to the expression (12), as the following reasoning will show. Suppose the time-coordinates are so chosen that

$$(x_0-a)^2+(y_0-b)^2+(z_0-c)^2+(l_0-\lambda)^2=0,$$

i.e. $c(t_0-t')=-r$,

or $t=t_0+\frac{r}{c}$.

We are in fact estimating the effect at the external point r/c seconds after the electron had been in the position (x_0, y_0, z_0).

Then, since

$$R^2=(x-a)^2+(y-b)^2+(z-c)^2+(l-\lambda)^2+[(x-a)w_1 +(y-b)w_2+(z-c)w_3+(l-\lambda)w_4],$$

we can, denoting by R' the four-vector with the components $\{(x-a), (y-b), (z-c), (l-\lambda)\}$,

write $$R^2=R'^2+(R'w)^2,$$

where $(R'\omega)$ denotes the scalar product of the four-vectors R' and ω.

With the above assumption, we have $R'=0$,

$$\therefore\ |R| = |(R'\omega)| =\frac{u_1(x-a)+u_2(y-b)+u_3(z-c)-r}{\sqrt{(1-u^2)}},$$

$\therefore$ We can write

$$[\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4]=\frac{e\omega}{(R'\omega)}=\frac{r}{r(1-u_r)}\,[u_1, u_2, u_3, i].$$

Using the ordinary time-coordinate, we have

$$\phi=\frac{e}{r\left(1-\frac{u_r}{c}\right)}, \ \mathrm{F. G. H}=\frac{e(u_1, u_2, u_3)/c}{r\left(1-\frac{u_r}{c}\right)} \qquad \ldots(12)$$

This result has been obtained in various ways by Herglotz[21], Sommerfeld[22], and other workers. Sommerfeld effects the integration of equation (10 a), with the aid of Cauchy's law of residues, and confirms the result (previously obtained by Herglotz),

$$\mathbf{a}=\frac{e\omega}{R'\omega} \qquad \ldots(11')$$

But a comparison of the methods of arriving at the two formulae will show that the expression (11′) is but a partial statement of the result, it being assumed from the very beginning that the time-coordinates are separated by the interval r/c, where r=three-dimensional distance between the points. The result $\mathbf{a}=\frac{e\omega}{R}$ is perfectly general, and in full agreement with the requirements and the spirit of the principle of relativity. This reduces to the expression (11′), when for the purpose of forming an idea of the result in three-dimensions, we make the particular assumption just mentioned about the time-coordinates. Hence it is apparent that when we apply the result to the determination of the magnetic and electric forces, and the ponderomotive force, we must use the expression (11), and not (11′).

9. *The Ponderomotive Force on an Electron due to the field produced by the motion of another electron.*

In § 6 we investigated the action of a point-charge on another charge; in the present section we shall investigate the action of an electron (B) [coordinates of centre (a, b, c, λ), velocity components (v_1, v_2, v_3)] upon another electron A [charge e, coordinates of centre $(x, y, z, 1)$, velocities (u_1, u_2, u_3)].

[20] *L'Eclairage Electrique*, vol xvi, 1898; Wiechert, *Ann. d. Physik*, vol. iv.

[21] Herglotz, *Gött. Nach.* Heft 6. (1940).
[22] Sommerfeld, *Ann. d. Physik*, vol. xxxiii. p. 666.

The components of the ponderomotive force upon a point (x', y', z', l') of the electron A are given by the four components of the matrix

$$\rho_0 \begin{vmatrix} \omega_1 & \omega_2 & \omega_3 & \omega_4 \\ \frac{\partial}{\partial x'} & \frac{\partial}{\partial y'} & \frac{\partial}{\partial z'} & \frac{\partial}{\partial l'} \\ \mathbf{a}'_1 & \mathbf{a}'_2 & \mathbf{a}'_3 & \mathbf{a}'_4 \end{vmatrix},$$

where $\mathbf{a}' = \frac{e'\omega'}{R'}$,

$$R'^2 = (x-a)^2 + (y-b)^2 + (z-c)^2 + (l-\lambda)^2 + [\omega_1'(x-a) + \omega_2'(y-b) + \omega_3'(z-c) + \omega_4'(l-\lambda)]^2,$$

and $[\omega_1', \omega'_2, \omega_3', \omega_4'] = \frac{1}{\sqrt{(1-\nu^2)}} [\nu_1, \nu_2, \nu_3, i]$.

The total force is obtained by integrating each of these four expressions over the whole volume of the electron A. The X-component of force

$$\rho_0 = \left[\frac{\partial\phi'}{\partial X'} - \frac{d}{d\tau'}\left(\frac{\partial\phi'}{\partial\omega'}\right)\right],$$

where $d\tau' = \sqrt{(1-u^2)} \cdot dt$,

and $\omega_1 = \frac{\partial X'}{\partial\tau} = \frac{\partial X'}{\partial\tau}$,

for all points of the electron move with the same velocity,

and $\phi' = \frac{e'}{R'} [\omega_1\omega_1' + \omega_2\omega_2' + \omega_3\omega_3' + \omega_4\omega_4']$.

The total force

$$X = \iiint \rho_0 \left[\frac{\delta\phi}{\delta x'} - \frac{d}{d\tau}\left(\frac{\delta\phi}{\delta\omega_1}\right)\right] d\Omega;$$

$d\Omega$ being the contents of the normal section of the cylinder

$$(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 + (l-l_0)^2 + [x'-x_0)w_1 + (y'-y_0)w_2 + (z'-z_0)w_3 + (l'-lo)w_4]^2 = r^2. \quad (13)$$

It can be easily proved that

$$\frac{\partial\phi}{\partial x'}\left(\frac{1}{R'}\right) = \frac{\partial}{\partial x_0}\left(\frac{1}{R}\right)$$

for the points x_0 and x are rigidly connected. Accordingly

$$X = \iiint \left[\rho_0 \left\{\frac{\partial\phi'}{\partial x_0} - \frac{d}{d\tau}\left(\frac{\partial\phi'}{\partial\omega_1}\right)\right\}\right] d\Omega.$$

Let $\int \rho_0 \phi' d\Omega = \Phi$.

Then $X = \frac{\partial\phi}{\partial x_0} - \frac{d}{d\tau}\left(\frac{\partial\phi}{\partial\omega_1}\right)$.

To evaluate Φ we need only find out the value of the integral

$$I = \iiint \frac{d\Omega}{R'}.$$

Introducing the Lorentz-transformation, in which the axis of motion of the cylinder (13) becomes the new time-axis, we have now

$$I = \iiint \frac{d\xi d\eta d\zeta}{R'}$$

over the volume $(\xi-\xi_0)^2 + (\eta-\eta_0)^2 + (\zeta-\zeta_0)^2 \gtreqless r^2$; and R' is expressed in terms of the new-coordinate system. Let $(\xi', \eta', \zeta', \nu')$ be the new coordinates of B.

Then

$$R'^2 = (\xi-\xi')^2 + (\eta-\eta')^2 + (\zeta-\zeta')^2 + (\nu-\nu')^2 + [(\xi-\xi')\omega_1'' + (\eta-\eta')\omega_2'' + (\zeta-\zeta')\omega_3'' + (\nu-\nu')\omega_4'']^2,$$

where $(\omega_1'', \omega_2'', \omega_3'', \omega_4'')$ are the direction cosines of the axis of B in the new system. R' is therefore of the form

$$R'^2 = A\xi^2 + B\eta^2 + C\zeta^2 + 2H\xi'\eta' + 2G\eta'\zeta' + 2F\eta'\zeta' + 2U\xi' + 2V\eta' + 2W\zeta' + D.$$

Let (R) be the same function of (ξ_0, η_0, ζ_0) i.e. R is now the perpendicular distance of the centre of A from the axis of B.

Then it can be proved that approximately

$$I = \frac{\Omega}{R}\left[1 - \frac{(2-\omega_4'')}{R} r^2 + \quad . \quad . \quad . \quad . \quad . \quad \right].$$

Neglecting terms of higher order than the first, we have

$$I = \frac{\Omega}{R}.$$

(In view of the fact that the radius of the electron is extremely small, the second term must be infinitesimal of a higher order compared with the first).

Therefore as a first approximation,

$$\Phi = \frac{ee'}{R}(\omega_1\omega_1' + \omega_2\omega_2' + \omega_3\omega_3' + \omega_4\omega_4').$$

$$\therefore \left.\begin{aligned} X &= \frac{\partial\Phi}{\partial x} - \frac{d}{d\tau}\left(\frac{\partial\Phi}{\partial\omega_1}\right) \\ Y &= \frac{\partial\Phi}{\partial y} - \frac{d}{d\tau}\left(\frac{\partial\Phi}{\partial\omega_2}\right) \\ Z &= \frac{\partial\Phi}{\partial z} - \frac{d}{d\tau}\left(\frac{\partial\Phi}{\partial\omega_3}\right) \\ L &= \frac{\partial\Phi}{\partial l} - \frac{d}{d\tau}\left(\frac{\partial\Phi}{\partial\omega_4}\right) \end{aligned}\right\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

dropping the subscripts 0, (x, y, z, l) now denoting the coordinates of the centre.

10. *Laws of Electrodynamical Action.*

We shall now reduce the Lagrangian function to three-dimensions. We have

$$\Phi = \frac{ee'(\omega_1\omega_1' - \omega_2\omega_2' + \omega_3\omega_3' + \omega_4\omega_4')}{\{(x-a)^2 + (y-b)^2 + (z-c)^2 + (l-\lambda)^2 + [(x-a)\omega_1' + \ldots]^2\}^{\frac{1}{2}}}.$$

Putting $(x-a)^2 + (y-b)^2 + (z-c)^2 + (l-\lambda)^2 = 0$ just as we did in the interpretation of the potential-four-vector, we have

$$\Phi = \frac{ee'(u_1v_1 + u_2v_2 + u_3v_3 - 1)}{r(1-v_r)\sqrt{(1-u^2)}} \quad \ldots (15)$$

with the same interpretation for r and v_r as before.

Excepting for the factor $[(1-v_r)\sqrt{(1-u^2)}]$ in the denominator, the form for P is identical with that assumed by Clausius for explaining the laws of electrodynamic action. The occurrence of these terms need not cause us any confusion; following in the wake of Clausius, we can easily prove that this formula leads to the laws of electrodynamical action just as well as any one of the formulae mentioned in the introduction. We have to take terms up to the second order, and instead of using r, we shall have to introduce the instantaneous distance r' which differs from $r\ (1-v_r)$ by terms of second order only. The second order terms arising out of $r\ (1-v_r)$ and $\sqrt{(1-u^2)}$ affect only one electron; while the term $(u_1v_1+u_2v_2+u_3v_3)$ affects both of them. Remembering that current consists of equal quantities of positive and negative charges moving in opposite directions, there will be no difficulty in realizing that in the final process of summation, terms affecting only one electron would cancel out, and only terms involving both of the electrons would remain in the final result. For further particulars, I would refer the reader to the above-mentioned memoir of Clausius's where the whole thing is worked out in a most elaborate and convincing manner.

11.

While the main object which I had in view when the work was undertaken has been achieved, viz., the deduction of the laws of electrodynamical action between two closed currents from the theory of electrons, I wish to point out certain other consequences to which this investigation may lead. With the help of Minkowski's four-dimensional analysis, I have succeeded in recasting the important result of Lienard and Wiechert (on the field produced by a moving electron) in an entirely novel form, and as I believe the only form consistent with the principle of relativity. The potential-four-vector has been proved to be equivalent to $(e\omega/R)$, where e=total charge, w=velocity-four-vector of the electron, and R is the four-dimensional perpendicular distance of the external point from the axis of motion of the electron. By applying the theorem in this simple form to Lorentz's equations for the ponderomotive force acting on an electron, it has been found possible to deduce a Lagrangian function controlling the motion of two electrons round each other. It has been shown that for small velocities, the result is practically identical with that tentatively assumed by Clausius in 1880 for explaining the laws of electrodynamical action on the atomistic hypothesis. There is one important distinction to which attention should be drawn.

In the usual form of Lagrangian equations of motion, we express the force X in the form

$$X=\frac{\partial\phi}{\partial x}-\frac{d}{d\tau}\left(\frac{\partial\phi}{\partial\frac{dx}{dt}}\right).$$

But here we have

$$X=\frac{\partial\phi}{\partial x}-\frac{d}{d\tau}\left(\frac{\partial\phi}{\partial\frac{dx}{d\tau}}\right),$$

i.e. in place of time t, we have to use the proper-time τ, where $d\tau=\sqrt{(1-u^2)}\ dt$.

The name proper-time for the function τ, suggests that it has some special relation with the time-coordinate, whereas in fact it is perfectly symmetrical, and similarly related to each of the four coordinates. To dispel any such false notion, it is now usual to designate $d\tau$ as an element of length of the world-line of motion. Thus

$$d\tau=ds=\sqrt{dt^2-dx^2-dy^2-dz^2},$$

and $(\omega_1,\ \omega_2,\ \omega_3,\ \omega_4)$ becomes $\sqrt{-1}$ times the direction cosines of the element dS.

In a system consisting of two electrons only, the forces controlling the motion are due to electronic attraction only; the gravitational field, being 10^{-42} times smaller than the electronic field, can be entirely neglected. Following Minkowski,[23] the equations of motion can be written in the forms:

$$\left.\begin{aligned}
c^2m_0\frac{d^2x}{ds^2}&=\frac{\partial\phi}{\partial x}-\frac{d}{ds}\left(\frac{\partial\phi}{\partial\frac{dx}{ds}}\right) & m'\frac{d^2a}{ds'^2}&=\frac{\partial\phi'}{\partial a}-\frac{d}{ds'}\left(\frac{\partial\phi'}{\partial\frac{da}{ds'}}\right)\\
c^2m_0\frac{d^2y}{ds^2}&=\frac{\partial\phi}{\partial y}-\frac{d}{ds}\left(\frac{\partial\phi}{\partial\frac{dy}{ds}}\right) & m'\frac{d^2b}{ds'^2}&=\frac{\partial\phi'}{\partial b}-\frac{d}{ds'}\left(\frac{\partial\phi'}{\partial\frac{db}{ds'}}\right)\\
c^2m_0\frac{d^2z}{ds^2}&=\frac{\partial\phi}{\partial z}-\frac{d}{ds}\left(\frac{\partial\phi}{\partial\frac{dz}{ds}}\right) & m'\frac{d^2c}{ds'^2}&=\frac{\partial\phi'}{\partial c}-\frac{d}{ds'}\left(\frac{\partial\phi'}{\partial\frac{dc}{ds'}}\right)\\
c^2m_0\frac{d^2l}{ds^2}&=\frac{\partial\phi}{\partial l}-\frac{d}{ds}\left(\frac{\partial\phi}{\partial\frac{dl}{ds}}\right) & m'\frac{d^2\lambda}{ds'^2}&=\frac{\partial\phi'}{\partial\lambda}-\frac{d}{ds'}\left(\frac{\partial\phi'}{\partial\frac{d\lambda}{ds'}}\right)
\end{aligned}\right\}\quad\ldots(16)$$

These equations are a particular case of the general equations of motion of an electron

$$\left.\begin{aligned}
\frac{m_0c^2}{e}\frac{d^2x}{d\tau^2}&=\qquad\omega_2f_{12}+\omega_3f_{13}+\omega_4f_{14}\\
\frac{m_0c^2}{e}\frac{d^2y}{d\tau^2}&=\omega_1f_{21}\qquad+\omega_3f_{23}+\omega_4f_{24}\\
\frac{m_0c^2}{e}\frac{d^2z}{d\tau^2}&=\omega_1f_{31}+\omega_2f_{32}\qquad+\omega_4f_{34}\\
\frac{m_0c^2}{e}\frac{d^2l}{d\tau^2}&=\omega_1f_{41}+\omega_2f_{42}+\omega_3f_{43}
\end{aligned}\right\}\quad\ldots(17)$$

These equations can be deduced from the Principle of Least Action in the following manner. The ordinary form of the Principle of Least Action is

$$\delta\int(T-V)\ dt=0 \qquad\ldots(18)$$

Instead of dt we write $d\tau=\sqrt{dt^2-dx^2-dy^2-dz^2}$, and for T we write m_0c^2, where m_0=rest-mass of the electron.

We have then

$$\delta V=X\delta x+Y\delta y+Z\delta z+L\delta l,$$

where $(X,\ Y,\ Z,\ L)$ are the components of the Pondermotive

[23] Minkowski, loc. cit. Appendix.

Force-four-vector, $(\delta x, \delta y, \delta z, \delta l)$ are the variational displacements.

Instead of the ordinary form, we have now

$$\delta\int m_0c^2 d\tau - \int \delta V\, d\tau = 0 \qquad \ldots(18')$$

Now $d\tau = -(\omega_1 dx + \omega_2 dy + \omega_3 dz + \omega_4 dl)$,

and $\delta V d\tau = -[X\delta x + Y\delta y + Z\delta z + L\delta l]\, ds$

$$= -e[f_{12}(\delta x dy - dx\delta y) + f_{23}(\delta y dz - dy\delta z) + f_{31}(\delta z dx - dz\delta x) + f_{14}(\delta x dl - dx\delta l) + f_{24}(\delta y dl - dy\delta l) + f_{34}(\delta z dl - dz\delta l)].$$

Now we shall prove an auxiliary theorem[24]; the ("X, Y, Z, L") used in this proof have no connexion with the force-components.

We have

$$\delta\int X dx + Y dy + Z dz + L dl$$
$$= \Sigma\int \delta X dx + \int X \delta dx.$$
$$= \sum\int\left(\frac{\partial X}{\partial x}\delta x + \frac{\partial X}{\partial y}\delta y + \frac{\partial X}{\partial z}\delta z + \frac{\partial X}{\partial l}\delta l\right) dx +$$
$$+ \sum\int\left[X\left(\frac{\partial \delta x}{\partial x}dx + \frac{\partial \delta x}{\partial y}dy + \frac{\partial \delta x}{\partial z}dz + \frac{\partial \delta x}{\partial l}\right) dl\right].$$

After partial integration, the second term equals

$$\Sigma X\delta x\Big|_{\text{final}}^{\text{Initial}} - \sum\int\left(\frac{\partial X}{\partial x}dx + \frac{\partial X}{\partial y}dy + \frac{\partial X}{\partial z}dz + \frac{\partial X}{\partial l}dl\right)\delta x.$$

Hence $\delta\int X dx + Y dy + Z dz + L dl$

$$= X\delta x + Y\delta y + Z\delta z + L\delta l\Big|_{\text{final}}^{\text{initial}} + \int\left(\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}\right)$$

$(\delta x dy - dx\delta y) + 5$ other similar terms.

Now for (X, Y, Z, L) substitute m_0c^2 $(\omega_1, \omega_2, \omega_3, \omega_4)$ and and let us denote

$$\left(\frac{\partial \omega_k}{\partial x_h} - \frac{\partial \omega_h}{\partial x_k}\right) \text{ by } \Omega_{hk}.$$

Then we have

$$\delta\int m_0c^2 ds = \delta\int m_0c^2(\omega_1 dx + \omega_2 dy + \omega_3 dz + \omega_4 dl$$
$$= m_0c^2(\omega_1\delta x + \omega_2\delta y + \omega_3\delta z + \omega_4\delta l)\Big|_{\text{final}}^{\text{initial}}$$
$$+ \int[\Omega_{12}(\delta x dy - dx\delta y) + \Omega_{23}(\delta y dz - dy\delta z) + \Omega_{31}(\delta z dx - dz\delta x) + \Omega_{14}(\delta x dl - dx\delta l) + \Omega_{24}(\delta y dl - dy\delta l) + \Omega_{34}(\delta z dl - dz\delta l)].$$

Putting the first term $=0$ as usual, we have from equation (18')

$$\int[(m_0c^2\Omega_{12} + ef_{12})\,(\delta x dy - dx\delta y) + 5 \text{ other similar terms}] = 0.$$

The six-components of the six-vector $(\delta s \times ds)$ are not independent, hence we cannot put their coefficients individually $=0$. If this were possible we would have obtained the system of equations

$$-\frac{m_0c^2}{e} = \frac{f_{12}}{\Omega_{12}} = \frac{f_{23}}{\Omega_{23}} = \frac{f_{31}}{\Omega_{31}} = \frac{f_{14}}{\Omega_{14}} = \frac{f_{24}}{\Omega_{24}} = \frac{f_{34}}{\Omega_{34}},$$

since (dx, dy, dz, dl) represent the actual displacement, $(\delta x, \delta y, \delta z, \delta l)$ the variational displacements.

We shall have to collect the coefficients of $(\delta x, \delta y, \delta z, \delta l)$ separately and put them individually equal to zero. In this way we obtain the four equations

$$\left.\begin{aligned} -\frac{m_0c^2}{e} &= \frac{f_{12}\omega_2 + f_{13}\omega_3 + f_{14}\omega_4}{\Omega_{12}\omega_2 + \Omega_{13}\omega_3 + \Omega_{14}\omega_4} = \frac{f_{21}\omega_1 + f_{23}\omega_3 + f_{24}\omega_4}{\Omega_{21}\omega_1 + \Omega_{23}\omega_3 + \Omega_{24}\omega_4} \\ &= \frac{f_{31}\omega_1 + f_{32}\omega_2 + f_{34}\omega_4}{\Omega_{31}\omega_1 + \Omega_{32}\omega_2 + \Omega_{34}\omega_4} = \frac{f_{41}\omega_1 + f_{42}\omega_2 + f_{43}\omega_3}{\Omega_{41}\omega_1 + \Omega_{42}\omega_2 + \Omega_{43}\omega_3} \end{aligned}\right\} \qquad (17')$$

Of these, only three are independent.
It is easy to see that

$$m_0c^2[\omega_2\Omega_{12} + \omega_3\Omega_{13} + \omega_4\Omega_{14}] = m_0c^2\,\frac{d^2x}{ds^2},$$

for
$$\omega_2\Omega_{12} + \omega_3\Omega_{13} + \omega_4\Omega_{14} = \omega_2\left(\frac{\partial\omega_2}{\partial x} - \frac{\partial\omega_1}{\partial y}\right) + \omega_3\left(\frac{\partial\omega_3}{\partial x} - \frac{\partial\omega_1}{\partial x}\right) + \omega_4\left(\frac{\partial\omega_4}{\partial x} = \frac{\partial\omega_1}{\partial l}\right)$$
$$= \tfrac{1}{2}\frac{\partial}{\partial x}\left[\omega_1^2 \times \omega_2^2 + \omega_3^2 + \omega_4^2\right] - \left(\omega_1\frac{\partial}{\partial x} + \omega_2\frac{\partial}{\partial y} + \omega_3\frac{\partial}{\partial z} + \omega_4\frac{\partial}{\partial l}\right)\omega_1$$
$$= -\frac{d\omega_1}{ds} = -\frac{d^2x}{ds^2}$$

for $\omega_1^2 + \omega_2^2 + \omega_3^2 + \omega_4^2 = -1$, $\therefore$ the first term$=0$.

The system of equations (17') are thus practically identical with the equations (17) but for practical purposes this form may be more convenient than the Minkowskian form.

The six-vector Ω may be styled as the "acceleration" six-vector, $(\Omega_{23}, \Omega_{31}, \Omega_{12})$ being connected with the three components of rotation, and $(\Omega_{14}, \Omega_{24}, \Omega_{34})$ with the three components of acceleration

$$\left(\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}, \frac{d^2z}{dt^2}\right).$$

In conclusion, I wish to express my thanks to Prof. D. N. Mallik, and my friend Mr. Satyendra Nath Basu for their kind help and encouragement.[25]

[24] Vide Cunninghum, Principle of Relativity, Chap VIII.

[25] The paper was communicated about two years ago, but owing to irregularities of the mail service caused by the war, the publication has been rather delayed. Meanwhile much work has been published on the subject, especially several important papers by Crehore in the 'Physical Review'. The author takes this opportunity of expressing his regret that he has not been able to compare his results with those obtained by Crehore and other workers.