1. ON MAXWELL'S STRESSES*

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1. Maxwell[1] has shown that the mechanical action between two electrical systems at rest can be accounted for by assuming the existence of certain stresses distributed over a surface completely enclosing one of the systems. If ψ be the potential at any point due to the whole system, the X-component of the mechanical force on one of the systems can be shown to be

$$F_x = \frac{1}{4\pi} \iiint \frac{\partial \psi}{\partial x} \nabla^2 \psi \; dx\, dy\, dz, \tag{1}$$

where the integration extends over the space occupied by the first system.

2. If the force is really due to the presence of stresses on a surface enclosing the first system, we have

$$F_x = \iint X_\nu dS = \iint (lX_x + mX_y + nX_z)\, dS, \tag{2}$$

where X_x, X_y, X_z, &c ... are the various surface-tractions, and (l, m, n) are the direction cosines of the normal to the surface.

By transforming expression (2), we obtain

$$F_x = \iiint \left(\frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z} \right) dx\, dy\, dz.$$

Since

$$\frac{1}{4\pi} \frac{\partial \psi}{\partial x} \nabla^2 \psi = \frac{\partial}{\partial x} \left[\frac{1}{8\pi} \left\{ \left(\frac{\partial \psi}{\partial x} \right)^2 - \left(\frac{\partial \psi}{\partial y} \right)^2 - \left(\frac{\partial \psi}{\partial z} \right)^2 \right\} \right] + \frac{\partial}{\partial y} \left[\frac{1}{4\pi} \frac{\partial \psi}{\partial x} \cdot \frac{\partial \psi}{\partial y} \right] + \frac{\partial}{\partial z} \left[\frac{1}{4\pi} \frac{\partial \psi}{\partial x} \cdot \frac{\partial \psi}{\partial z} \right],$$

we have, putting $\frac{\partial \psi}{\partial x} = X, \frac{\partial \psi}{\partial y} = Y, \frac{\partial \psi}{\partial z} = Z,$

$$\iiint \left\{ \frac{\partial}{\partial x} \left[X_x - \frac{1}{8\pi} \left(X^2 - Y^2 - Z^2 \right) \right] + \frac{\partial}{\partial y} \left[X_y - \frac{1}{4\pi} X Y \right] + \frac{\partial}{\partial z} \left[X_z - \frac{1}{4\pi} X Z \right] \right\} dx\, dy\, dz = 0 \tag{3}$$

Maxwell concludes from this that a system of stresses

$$X_x = \frac{1}{8\pi} (X^2 - Y^2 - Z^2), \quad Y_y = \frac{1}{8\pi} (Y^2 - Z^2 - X^2),$$

$$Z_z = \frac{1}{8\pi} (Z^2 - X^2 - Y^2), \quad X_y = \frac{1}{4\pi} XY,$$

$$Y_z = \frac{1}{4\pi} YZ, \quad Z_x = \frac{1}{4\pi} ZX, \tag{4}$$

distributed over the surface S account for the mechanical action quite satisfactorily, and therefore provide a concrete physical representation of the mechanism of electrostatic action.

3. But the expressions (4) are not complete solutions of the integral equation (3). Maxwell[2] himself points out that they can at best be regarded as a first step towards the solution of equation (3). Many investigators, including Sir J. J. Thomson[3], have pointed out that aether cannot possibly be at rest under these stresses. Lorentz[4] goes so far as to say that the stresses are simply mathematical fictions, which can be conveniently utilized for the calculation of radiation pressure and other allied phenomena. The object of the present paper is to show that the stresses cannot account for the energy of electrification, if the medium is to be regarded as an elastic solid.

4. The energy of a charged system can be expressed as a volume integral,

$$W = \frac{1}{8\pi} \iiint \left[\left(\frac{\partial \psi}{\partial x} \right)^2 + \left(\frac{\partial \psi}{\partial y} \right)^2 + \left(\frac{\partial \psi}{\partial z} \right)^2 \right] dx\, dy\, dz \tag{5}$$

Maxwell[5] states that the quantity W may be interpreted as the energy in the medium due to the distribution of stresses; but the statement is not proved. The only rational meaning which we can attach to this assertion is that the energy of electrification arises from the elastic displacement of aether particles. I am not aware whether any other interpretation has been or can be given to Maxwell's statement, but it has generally been taken in this sense, though Maxwell himself is rather vague on the point. We should naturally expect that energy calculated on this understanding would lead to the expression (5), but that this is not the case will be presently shown.

5. If u, v, w be the elastic displacements of a particle of the dielectric medium, the energy of deformation or the strain-energy function is

$$W' = \iiint\!\!\int_{\text{initial state}}^{\text{final state}} \rho\, (X\delta u + Y\delta v + Z\delta w)\, dx\, dy\, dz$$

$$+ \iint\!\!\int_{\text{initial state}}^{\text{final state}} (X_\nu \delta u + Y_\nu \delta v + Z_\nu \delta w)\, dS, \tag{6'}$$

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[1] Maxwell, 'Electricity and Magnetism', Vol. 1, Chap. V.

[2] Loc. cit. p. 165 et seq.

[3] Loc. cit. p. 165. footnote

[4] 'Theory of Electrons', p. 31.

[5] Electricity and Magnetism', p. 165.

and this can be shown to be equivalent to

$$\tfrac{1}{2}\iiint (X_x e_{xx} + Y_y e_{yy} + Z_z e_{zz} + X_y e_{xy} + Y_z e_{yz} + Z_x e_{zx})\, dx\, dy\, dz.$$

Assuming the aether to be isotropic and to behave as an elastic solid, we can put

$$\begin{bmatrix} e_{xx} \\ e_{yy} \\ e_{zz} \end{bmatrix} = \frac{1}{E}\begin{bmatrix} 1 & -\sigma & -\sigma \\ -\sigma & 1 & -\sigma \\ -\sigma & -\sigma & 1 \end{bmatrix} \begin{bmatrix} X_x \\ Y_y \\ Z_z \end{bmatrix},$$

and $e_{xy} = \dfrac{X_y}{\mu}$, $e_{yz} = \dfrac{Y_z}{\mu}$, $e_{zx} = \dfrac{Z_x}{\mu}$.

Then, after some calculation, the strain-energy function comes out to be

$$W' = \frac{1}{2}\,\frac{3(1+2\sigma)}{\epsilon(1+\sigma)} \iiint \left(\frac{R^2}{(8\pi^2)}\right)^2 dx\, dy\, dz \qquad (6)$$

It will thus be seen that if the stresses are really existent, and if they are amenable to the ordinary laws of elasticity, the strain-energy function, or the energy of elastic deformation of the medium, is $\dfrac{3}{2}\,\dfrac{(1+2\sigma)}{\epsilon\,(1+\sigma)}\left(\dfrac{(R^2)}{(8\pi^2)}\right)^2$ per unit volume. But this is very different from the theorem that the energy density per unit volume is $(R^2/8\pi)$ which is derived from electrostatic principles.

6. Since nothing definite is known about the elastic constants of aether, we cannot draw any conclusion from (6) about the energy distribution in aether. Maxwell's stresses are thus seen to fail to account for the energy of electrification, on the understanding that the medium behaves like an elastic solid.

7. It is well known that the energy-distribution theorem is proved on the basis of the empirical laws of electrostatics. No use is made of the stresses. The result is purely analytical, and says that if energy is distributed all over space as a continuous function with volume density $(R^2/8\pi)$, the total energy will come out to be the same as the total energy of electrification. The distinction between Maxwell's view of energy distribution as due to stresses (in the sense we have interpreted it) and the actual case can be better brought out if we adopt the following modified method of proving the energy-distribution theorem. Suppose we have an electrical system consisting of charged surfaces, and particles in a given configuration. The energy of electrification will be the same in whichever way we may bring about the final configuration. Suppose that, to start with, the charges and the charged surfaces were all at an infinite distance, and the given configuration is brought about by properly moving the charged surfaces and other discreet electrified particles. Then the energy of electrification is

$$\begin{aligned} W &= \Sigma \int e\,dV, \\ &= \Sigma \iint \sigma \left(\frac{\partial V}{\partial x}\,\delta x + \frac{\partial V}{\partial y}\,\delta y + \frac{\partial V}{\partial z}\,\delta z\right) dS, \\ &\quad + \Sigma \iiint \rho \left(\frac{\partial V}{\partial x}\,\delta x + \frac{\partial V}{\partial y}\,\delta y + \frac{\partial V}{\partial z}\,\delta z\right) dx\, dy\, dz, \end{aligned}$$

where σ is the surface density of electricity on a charged surface, and ρ is the volume density. Since $\dfrac{\partial V}{\partial x}$ is the x-component of electrical force on a surface, $\sigma\dfrac{\partial V}{\partial x}$ is the x-component of mechanical action per unit surface. Similarly, $\rho\dfrac{\partial V}{\partial x}$ is the x-component of mechanical force per unit volume of electrified particles. We can therefore put

$$W = \Sigma \iint (X_v \delta x + Y_v \delta y + Z_v \delta z) + \Sigma \iiint (X \delta x + Y \delta y + Z \delta z).$$

8. Comparing this expression for energy with the expression (6)

$$W' = \iint (X_v \delta u + Y_v \delta v + Z_v \delta w) + \iiint \rho (X \delta u + Y \delta v + Z \delta w), (6'')$$

we see that in the present case (X_v, Y_v, Z_v) are the components of surface-tractions on a charged surface, and (X, Y, Z) are the body-forces on electrified particles. The existence of these forces can be experimentally demonstrated, and they exist only in regions occupied by electricity; elsewhere they are nil. The energy of electrification is derived from the work done in the actual displacements $(\delta x, \delta y, \delta z)$ of these charged regions towards each other. On the other hand, (X_v, Y_v, Z_v) in (6″) are the tractions on a surface enclosing some of the charged regions, and δu, δv, δw are their elastic displacements. We may by special assumption identify the two systems of surface-tractions and body-forces, but the actual displacements $(\delta x, \delta y, \delta z)$ and the elastic displacements $(\delta u, \delta v, \delta w)$ cannot be identified in any way. The two expressions represent fundamentally different quantities.

9. The fact that radiant energy would exert a definite amount of pressure on material surfaces was first predicted by Maxwell on the hypothesis of dielectric stresses. Now that radiation pressure is an experimental fact, it has been supposed by some physicists that Maxwell's stresses must have a material existence. But it is well known that radiant energy can be deduced independently of the stresses. Bartoli has shown that the pressure of radiant energy can be deduced from thermodynamic principles. Planck[6] has deduced it from electrodynamical principles, assuming that the perfect reflector is a super-conductor of electricity. This is an ideal limiting case of the experimental fact that good conductors of electricity are also good reflectors of radiant energy. The electric vector accompanying a ray of light gives rise to a finite charge on the surface of the super-conductor and a finite current within the conductor. The charge exerts a negative pressure on the surface, while the current, in presence of the field of the magnetic vector accompanying the ray, produces a mechanical force in the contrary direction. The resultant of the two, when averaged statistically, yields the radiation pressure. How far these theories are consistent with the theory of stresses may form a subject for interesting investigations.

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[6] Planck, *Wärmestrahlung*, 2nd Edn., pp. 49 *et. seq.*