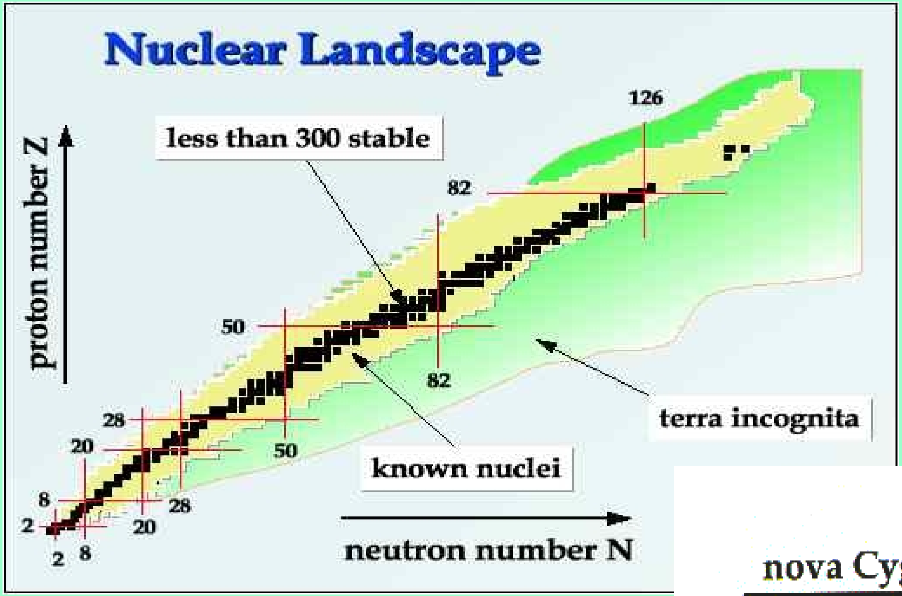
STATISTICAL SPECTROSCOPY METHOD FOR LEVEL DENSITIES AND GT- STRENGTHS IN NUCLEAR ASTROPHYSICAL APPLICATIONS

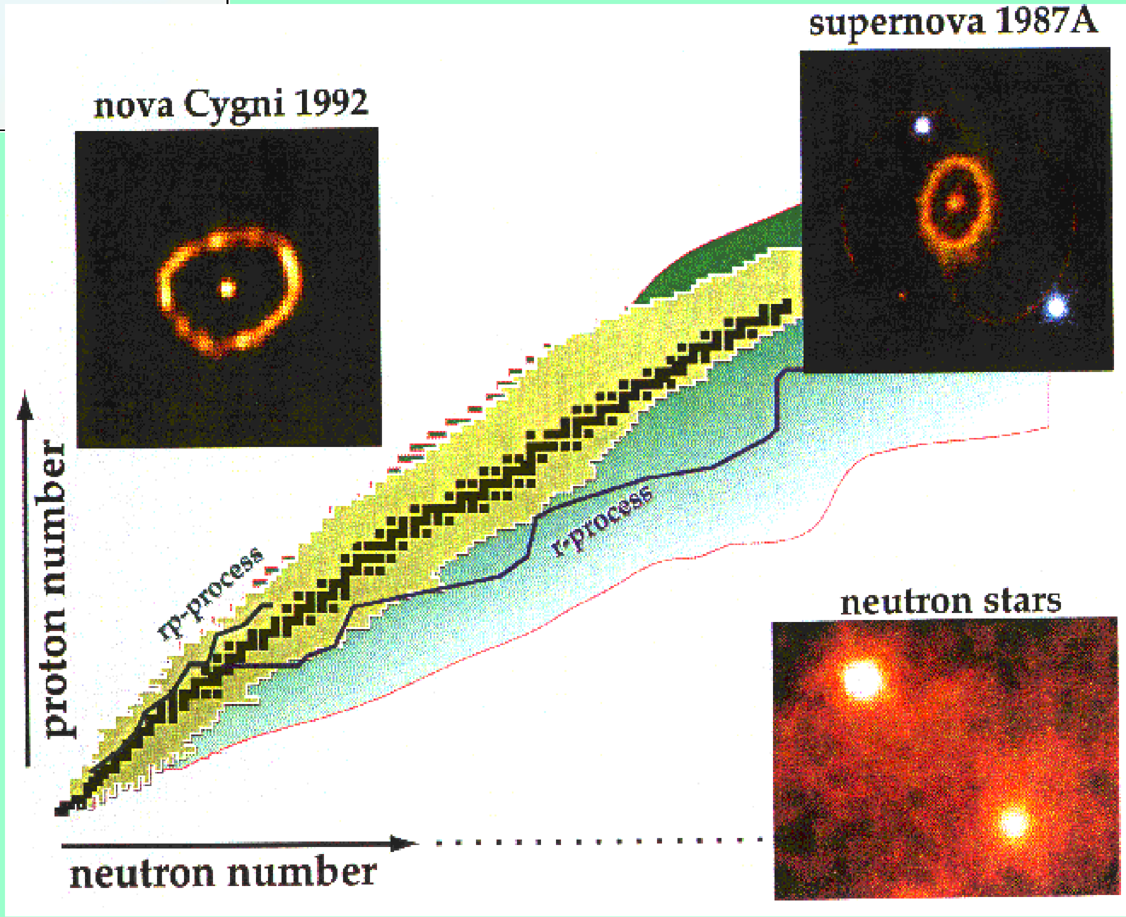
V.K.B. Kota

Theoretical Physics and Complex Systems Division

Physical Research Laboratory, Ahmedabad.



Nuclear Landscape
proton number Z
neutron number N
less than 300 stable
known nuclei
terra incognita
126
82
50
28
20
8
2
82
50
28
20
8
2



nova Cygni 1992
supernova 1987A
neutron stars
rp-process
r-process
proton number
neutron number

M.B. Aufderheide, I. Fushiki, S.E. Woosley and D.H. Hartmann, APJS **91**, 389 (1994); H. Schatz et al, Phys. Rep. **294**, 167 (1998); K.Langanke and G. Martinez-Pinedo, Rev. Mod. Phys. **75**, 819 (2003); J.Pruet and G.M. Fuller, APJS **149**, 189 (2003).

- Nuclear level densities and Gamow-Teller (GT) transition strengths for A >60 nuclei are two crucial nuclear structure inputs needed, for quantitative understanding of heavy nuclei nucleosynthesis via r- and rp-process and for explosive processes such as supernovae and X-ray bursts.

- Saha equation shows that the abundances of nuclear species depend, besides on Y_e, binding energy etc., also on the partition function G(Z,A,T) which is the Laplace transform of the level density.

- Weak-interaction processes, electron capture and beta decay, involving nuclei (with 40 < A < 90), influence strongly the late evolution stages of massive stars as they determine the core entropy and the electron to baryon ratio Y_e of the presupernovae star.

- Important for rp-process nuclear reaction network calculations above Z≥32 are the nuclear masses of the relevant neutron deficient isotopes, the proton-capture reaction rates as well as their inverse photo-disintegration rates, and finally the β-decay and electron-capture rates (for proton rich nuclei at least up to mass 110). First attempts to study nucleosynthesis in X-ray bursts was due to Wallace and Woosley in 1984.

- The r-process runs through very neutron rich, unstable nuclei, which are far from stability (most important are nuclei with N~82) and whose physical properties are often experimentally unknown. As relevant nuclear input, r-process simulations require neutron separation energies (i.e.masses), half-lives (determined by allowed GT transitions) and neutron-capture cross sections of the very neutron-rich nuclei on the various dynamical r-process paths.

In this talk we consider level densities and β-decay rates for A $\gtrsim$ 60 nuclei.

Methods for calculating level densities:

1. Fermi gas formulas with various modifications [Dilg et al, Nucl Phys. **A217**, 269 (1973); Iljinov et al, Nucl Phys. **A543**, 517 (1992); H.Nakamura and T. Fukahori, Phys. Rev. C **72**, 064329 (2005); D. Bucrescuand T. von Egidy, Phys. Rev. C **72**, 044311, 067304 (2005)].

2. Shell Model Monte Carlo (SMMC) method of Koonin et al [Y. Alhassid, G.F.Bertsch, L. Fang and S. Liu, Phys. Rev. C **72**, 064326].

3. Statistical Nuclear Spectroscopy method (SNSM) [French, Kota, Zelevinsky-----]

Methods for calculating β-decay rates:

1. FFN method -- data plus single particle model [G.M. Fuller, W.A.Flower, M.J. Newman, APJS **42**, 447 (1980); APJS **48**, 179 (1982); APJ **252**, 715 (1982); APJ **293**, 1 (1985); J. Pruet and G.M.Fuller, APJS **149**, 189 (2003)].

2. Shell Model [K. Langanke and G. Martinez-Pinedo, Rev. Mod. Phys. **75**, 819 (2003); Atomic Data and Nuclear Data Tables **79**, 1 (2001); J.L. Fisker et al, Atomic Data and Nuclear Data Tables **79**, 241(2001)].

3. Statistical Nuclear Spectroscopy method (SNSM): This is well suited as GT matrix elements involving excited states of the parent and daughter nucleus are needed for the temperatures involved in astrophysical processes[K. Kar, S. Sarkar, and A. Ray, Phys. Lett. **B261**, 217 (1991);Ap. J. **434**, 662 (1994)].

Plan of the talk

- **LAVEL DENSITIES AND BETA DECAY RATES: DEFINITIONS**
- **FOUNDATIONS OF SNS: TWO-BODY RANDOM ENSEMBLES AND MANY-BODY CHAOS**
- **SNSM FOR LEVEL DENSITIES: EXAMPLES**
- **SNSM FOR BETA DECAY RATES: EXAMPLES**
- **FUTURE OUTLOOK**

I. LEVEL DENSITIES AND β-DECAY RATES: DEFINITIONS

State density: $I^m(E) = \langle\langle \delta(H-E) \rangle\rangle^m = \sum_{\mathbf{m}} \langle\langle \delta(H-E) \rangle\rangle^{\mathbf{m}} = \sum_{\mathbf{m}} I^{\mathbf{m}}(E)$

Spin-cutoff factor: $\sigma_J^2(E) = \left\langle J_Z^2 \right\rangle^E$

Level density:

$$I_\ell(E,J) = \frac{2J+1}{\sqrt{8\pi}\sigma_J^3(E)} exp\left\{-\frac{(J+1/2)^2}{2\sigma_J^2(E)}\right\} I(E)$$

$$= \left\langle\left\langle \delta(H-E)\delta(J^2 - J(J+1)) \right\rangle\right\rangle^m /(2J+1) =$$

$$= \langle\langle \delta(H-E) \rangle\rangle^{m,J} /(2J+1)$$

$$\sum_{\mathbf{m}} \langle\langle \delta(H-E) \rangle\rangle^{\mathbf{m},J} /(2J+1) = \sum_{\mathbf{m}} I^{\mathbf{m},J}(E)$$

The decomposition into $I^{\mathbf{m}}(E)$, the partial densities, is exact. The **m** are eigenstates of the mean-field one-boby part of the nuclear hamiltonian. Also $I^{\mathbf{m},J}(E)$ are fixed-J partial densities and similarly one can define $I^{\mathbf{m},J,T}(E)$

The β-decay rate is the number of β-decays per second from a given initial state $|E_i\rangle$ of the parent nucleus to the final nuclear state $|E_f\rangle$ and the rate $\lambda(T)$ at finite temperature is the thermal average of the rates from all parent nucleus states. With Q the Q-value for β -decay from GS, $Q_i = Q + E_i$.

$$\text{GS half lives:} \quad t_{1/2}(GS) = \{6250 \ \ (s)\} \times \left\{ \int_0^Q \left[\left(\frac{g_A}{g_V}\right)^2 3\pounds \right] \left[\frac{\mathbf{I}^H_{\mathcal{O}(GT)}(E_{GS}, E_f)}{I^H(E_{GS})} \right] f(Z, Q - E_f) dE_f \right\}^{-1}$$

£ ~ 0.5-0.6 is the so called quenching factor which is required because the calculated (shell model) GT sum rule strength is always found to be larger than the observed strength.

$$\text{Bivariate GT strength density:} \quad \mathbf{I}^{m,m'}_{\mathcal{O}_{GT}}(E, E') = I^{m'}(E) \left| \langle E'm' \mid \mathcal{O}_{GT} \mid E, m \rangle \right|^2 I^m(E) = \left\langle\left\langle \mathcal{O}^\dagger_{GT} \delta(H - E') \mathcal{O}_{GT} \delta(H - E) \right\rangle\right\rangle^m$$

$$\lambda(T) = \frac{\ell n 2(s^{-1})}{6250} \left[\int e^{-E_i/k_B T} I^H(E_i) dE_i \right]^{-1} \times \left[\int dE_i e^{-E_i/k_B T} \left[\int_0^{Q_i} dE_f \left\{ \left(\frac{g_A}{g_V}\right)^2 3\pounds \right\} \mathbf{I}^H_{\mathcal{O}(GT)}(E_i, E_f) f(Z, T, Q_i - E_f) \right] \right]$$

For both β-decay and electron capture the phase space factor f and the coulomb factor F that f contains are known. An expressions for the chemical potential that enter into f is,

$$\mu_e = 1.11(\rho_7 Y_e)^{1/3} \left[1 + \left(\frac{\pi}{1.11}\right)^2 \frac{\bar{T}^2}{(\rho_7 Y_e)^{2/3}} \right]^{-1/3}$$

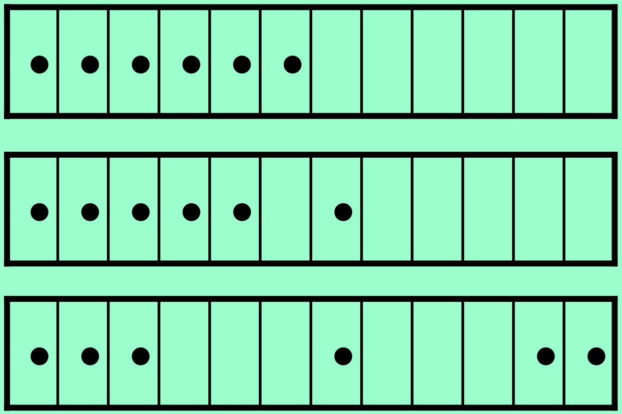
II. Foundations of SNS: Two-body random ensembles (TBRE) and Many-body chaos

- Embedded Gaussian orthogonal ensemble of two-body interactions: **EGOE(2)**
- Embedded Gaussian orthogonal ensemble of one plus two-body interactions: **EGOE(1+2)**

EGOE(2) for spinless fermion systems

N single particle states, m fermions

Basis states:
N=12, m=6



Number of basis states $d(N,m) = \binom{N}{m}$

d(12,6)=924, d(16,8)=12870

$$\hat{H}(2) = \sum_{i>j,k>l} H_{ijkl}\ a_k^\dagger a_l^\dagger a_j a_i$$

$$H_{ijkl} = \langle kl | H | ij \rangle$$

- H matrix in two-particle spaces is GOE
- Geometry gives H matrix in m-particle spaces
- Many m-particle matrix elements are zero
- There are correlations between m-particle matrix elements

J.B. French, S.S.M. Wong, Phys. Lett. B33, 447 (1970); B 35, 5 (1971).

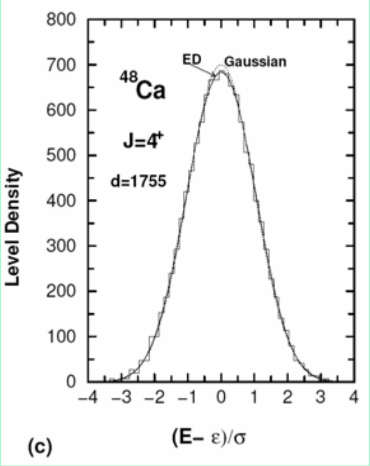
O. Bohigas, J. Flores, Phys. Lett. B34, 261 (1971); B35, 383 (1971).

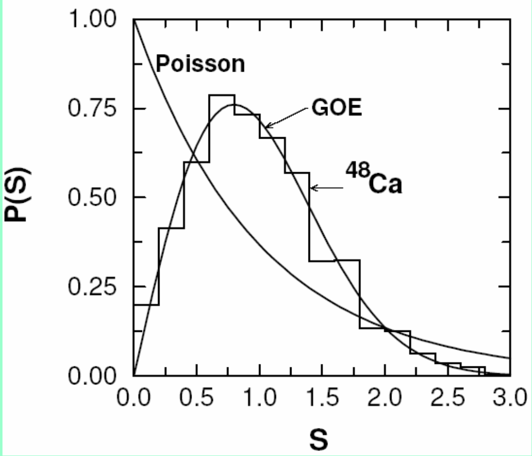
Results for EGOE(2)

i. Binary correlation approximation, applied in the dilute limit proves that density of states takes Gaussian form [K.K. Mon and J.B. French, Ann. Phys. (N.Y.) **95**, 90 (1975)].

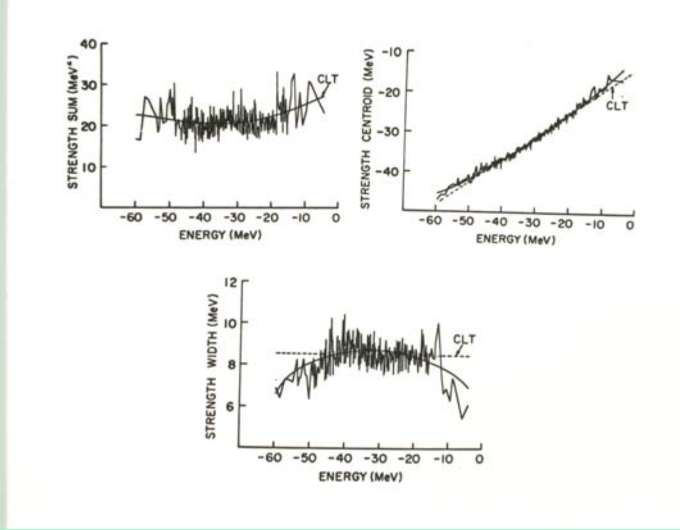
ii. Bivariate transition strength densities take in general bivariate Gaussian form [J.B. French, V.K.B. Kota, A. Pandey, S. Tomsovic, Phy. Rev. Lett. **58** (1987) 2400; Ann. Phys. (N.Y.) **181** (1988) 235].

iii. The transition from Semicircle to Gaussian takes at m=2k [L. Benet, T. Rupp, and H.A. Weidenmueller, Phys. Rev. Lett. **87**, 010601 (2001); Ann. Phys. (N.Y.) **292**, 67 (2001)].





$(2s,1d)^{6,J=2,T=0}$

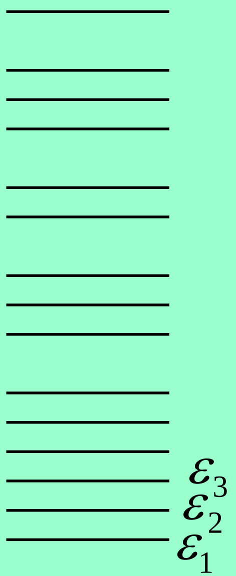


EGOE(1+2)

$$\hat{H} = \hat{h}(1) + \lambda \left\{ \hat{V}(2) \right\}$$

$$\hat{h}(1) = \sum_i \varepsilon_i \hat{n}_i, \quad \left\{ \hat{V}(2) \right\} \text{ is EGOE(2)}$$

$\hat{h}(1) \Leftrightarrow \varepsilon_i$ →
- fixed (TBRIM)
- random (TBRIM)
- drawn from GOE (RIMM)



Single particle spectrum
Δ is average spacing

EGOE(1+2) $\Leftrightarrow$ (m,N,λ/Δ)

$$\left| k \right\rangle = \sum_E C_k^E \left| E \right\rangle, \quad F_k(E) = \left| C_k^E \right|^2 \rho(E) \Leftrightarrow \text{ strength functions}$$

$$NPC(E) = \left\{ \sum_k \left| C_k^E \right|^4 \right\}^{-1}, \quad S^{\text{info}}(E) = -\sum_k \left| C_k^E \right|^2 \ln \left| C_k^E \right|^2$$

- NPC, S^{info} depend on density of states, strength functions and strength fluctuations
- NPC, S^{info} can be defined for transition strengths

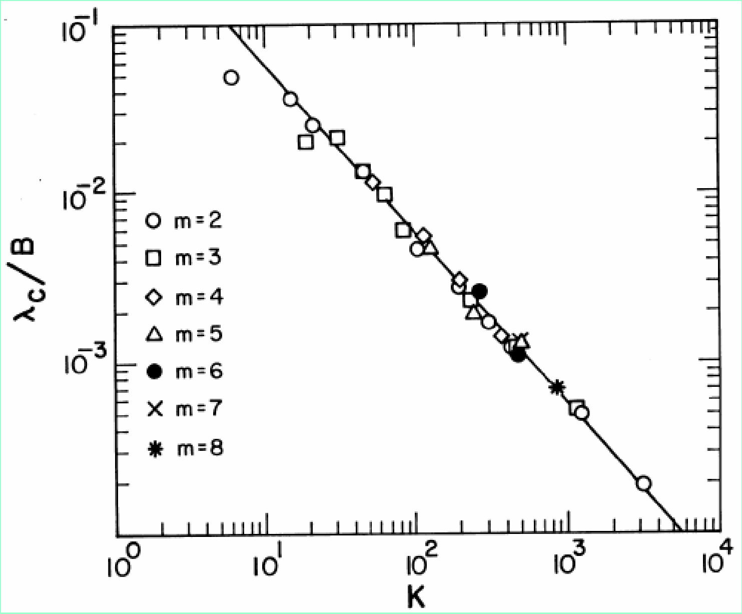
λ_c marker

$\lambda_c \propto$ spacing between directly coupled states $\Rightarrow \lambda_c \sim 1/(m^2N)$

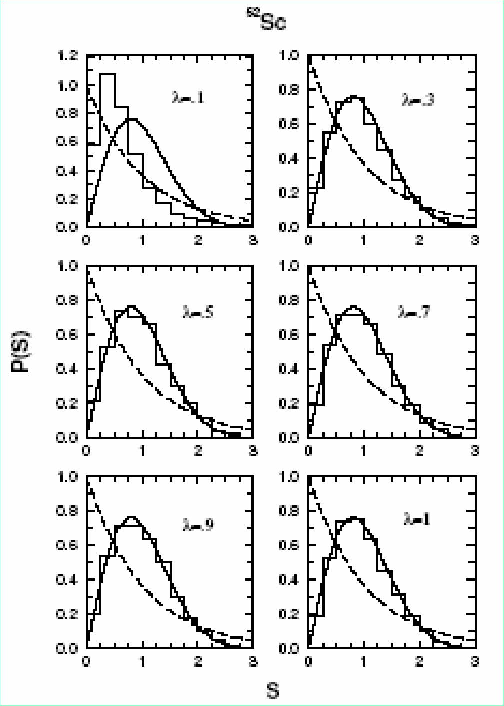
S. Aberg, Phys. Rev. Lett. **64**, 3119 (1990).

Ph. Jacquod and D.L. Shepelyansky, Phys. Rev. Lett. **79**, 1837 (1997).

J.M.G. Gomez, K. Kar, V.K.B. Kota, J. Retamosa and R. Sahu, Phys. Rev. C **64**, 034305 (2001).



K = Number of directly connected states
B = 2-particle spectrum span



J = 0^+, T = 5, d = 6107

λ_F marker

$\lambda_c < \lambda < \lambda_F$ region is called BW domain, $\lambda > \lambda_F$ region is called Gaussian domain

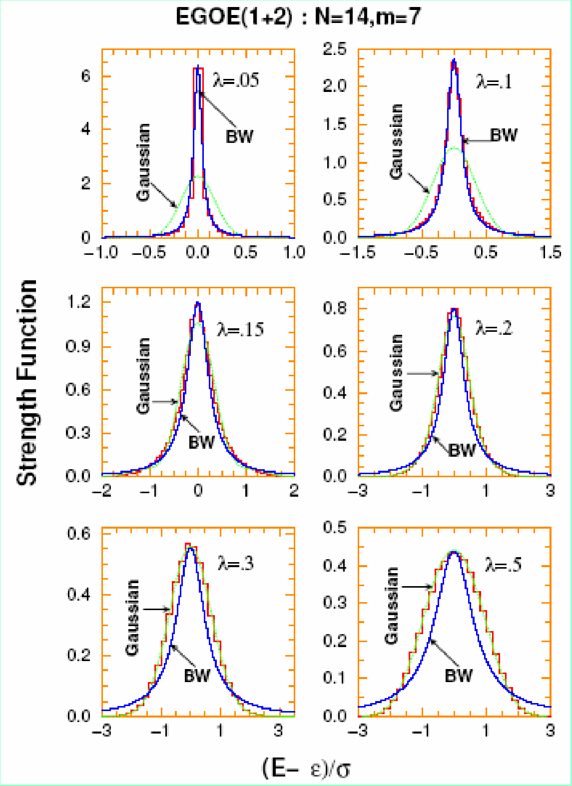
Arguments based on BW spreading widths give $\lambda_F \propto 1/\sqrt{m}$.

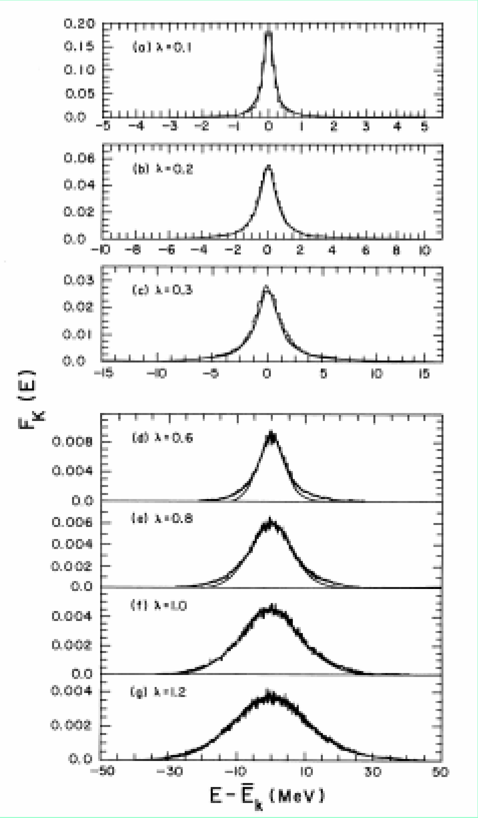
N. Frazier, B.A. Brown, and V. Zelevinsky, Phys. Rev. C **54**, 1665 (1996).

V.V. Flambaum and F.M. Izrailev, Phys. Rev. E **56**, 5144 (1997); Phys. Rev. E **61**, 2539 (2000).

V.K.B. Kota and R. Sahu, Phys. Rev. E **64**, 016219 (2001); preprint nucl-th/0006079

Ph. Jacquod and I. Varga, Phys. Rev. Lett. **89**, 134101 (2002).





J = 0, T = 0

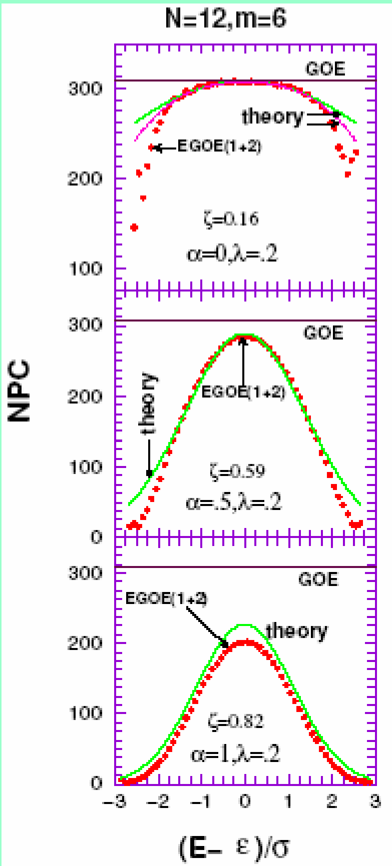
d = 839

^{28}Si

$$F_{BW}(E) = \frac{1}{2\pi} \frac{\Gamma}{(E - E_c)^2 + \Gamma^2/4}$$

$$F_G(E) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(E-E_c)^2}{2\sigma^2}}$$

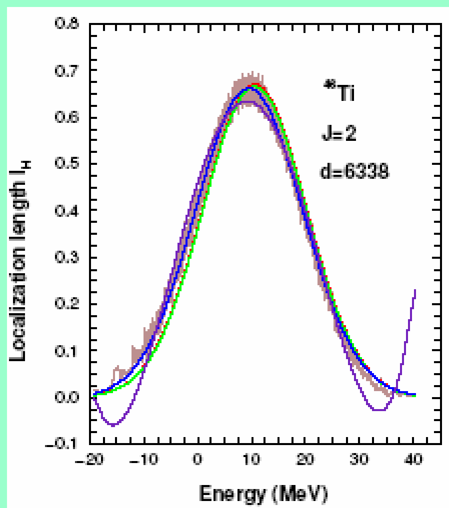
Number of principle components (NPC) and Information entropy (S^{info})

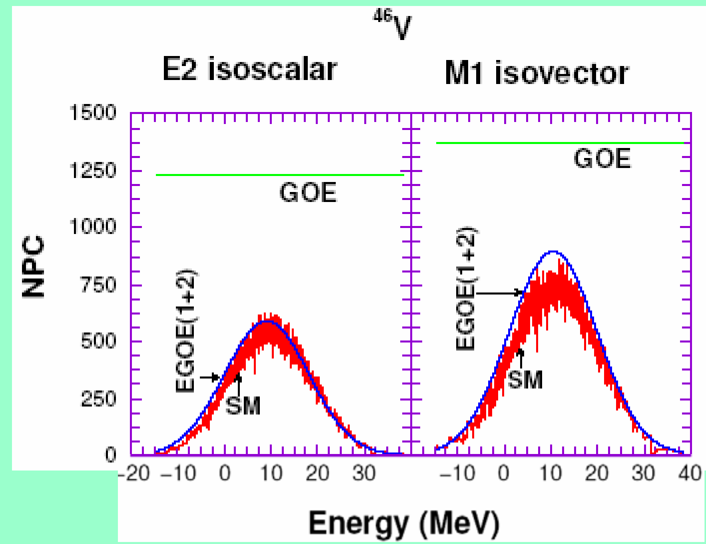


$$NPC(E) = (d/3)\sqrt{1-\zeta^4}\,\exp\left(-\frac{\zeta^2\hat{E}^2}{1+\zeta^2}\right)$$

$$\exp\left(S^{\text{info}}(E)\right) = (0.48d)\sqrt{1-\zeta^2}\,\exp\left(\frac{\zeta^2}{2}\right)\exp\left(-\frac{\zeta^2\hat{E}^2}{2}\right)$$

$$\zeta = \sqrt{1-\frac{\sigma^2_{off-diagonal}}{\sigma^2_{total}}}$$





814→3683 814→4105

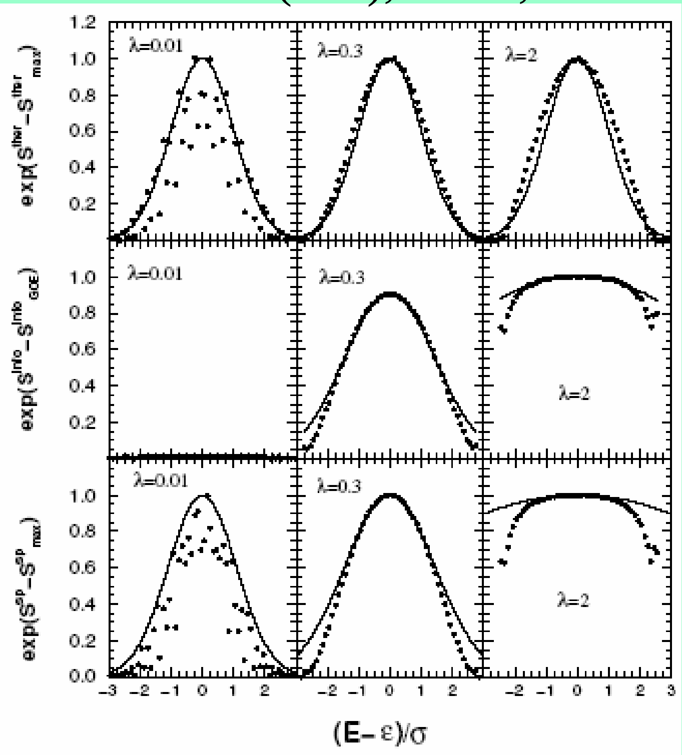
V.K.B. Kota and R. Sahu, Phys. Lett. **B429**, 1 (1998); Phys. Rev. E **64**, 016219 (2001).

J.M.G. Gomez, K. Kar, V.K.B. Kota, R.A. Molina, and J. Retamosa, Phys. Rev. C **69**, 057302 (2004).

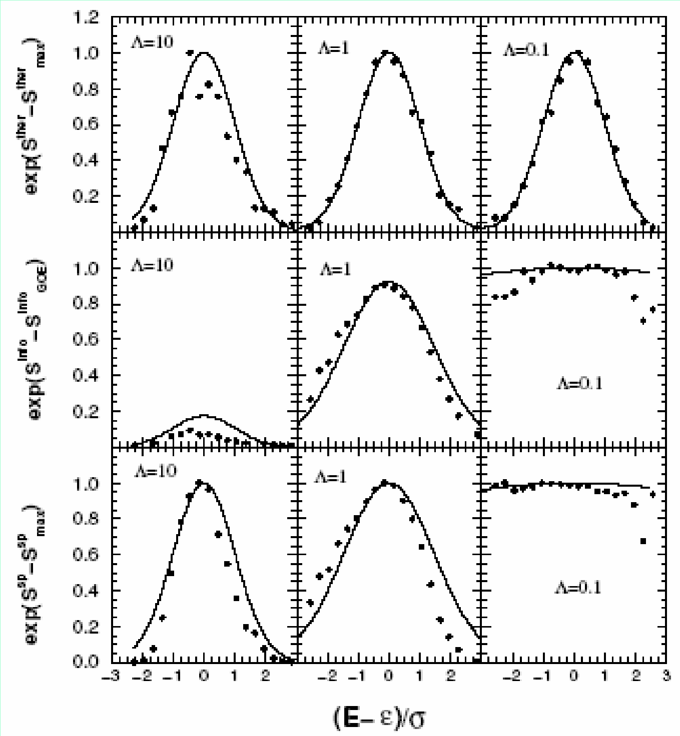
λ_t marker

Around λ_t different definitions of entropy [for example $S^{info}(E)$, thermodynamic entropy defined via $\rho^H(E)$, single particle entropy defined via occupation numbers], temperature etc. will coincide and also strength functions in h(1) and V(2) basis will coincide. Thus $\lambda \sim \lambda_t$ region is called the thermodynamic region.

EGOE(1+2); N=12,m=6



^{24}Mg (J = 0, T = 0)



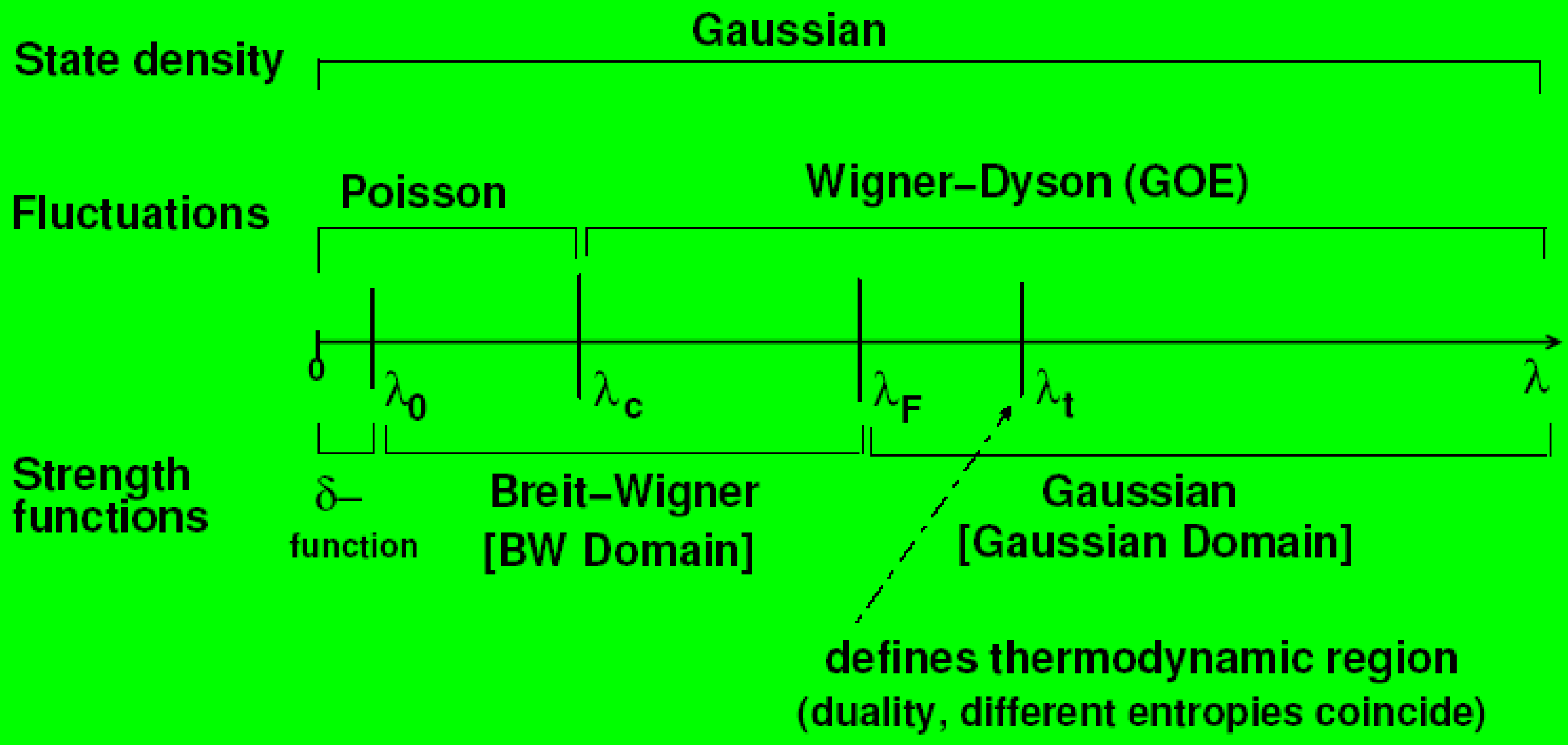
M. Horoi, V. Zelevinsky, and B.A. Brown, Phys. Rev. Lett. **74**, 5194 (1995).

V.K.B. Kota and R. Sahu, Phys. Rev. E **66**, 037103 (2002).

CHAOS MARKERS FOR EGOE(1+2)

$$H = \Delta\, h(1) + \lambda \{V(2)\}$$

$$(m, N, \Delta, \lambda)$$



III. SNSM FOR LEVEL DENSITIES: EXAMPLES

In order to extend the applicability of the TBRE Gaussian forms to large shell model spaces, it is essential to deal with partitioning of the m-particle spaces to subspaces [See Fig. 1]. Then,

$$m \to \sum S^{\pi}\ ;\ S^{\pi} \to \sum [\mathbf{m}];\ [\mathbf{m}] \to \sum \mathbf{m}\,;\ S = \sum m_{\alpha} s_{\alpha}$$

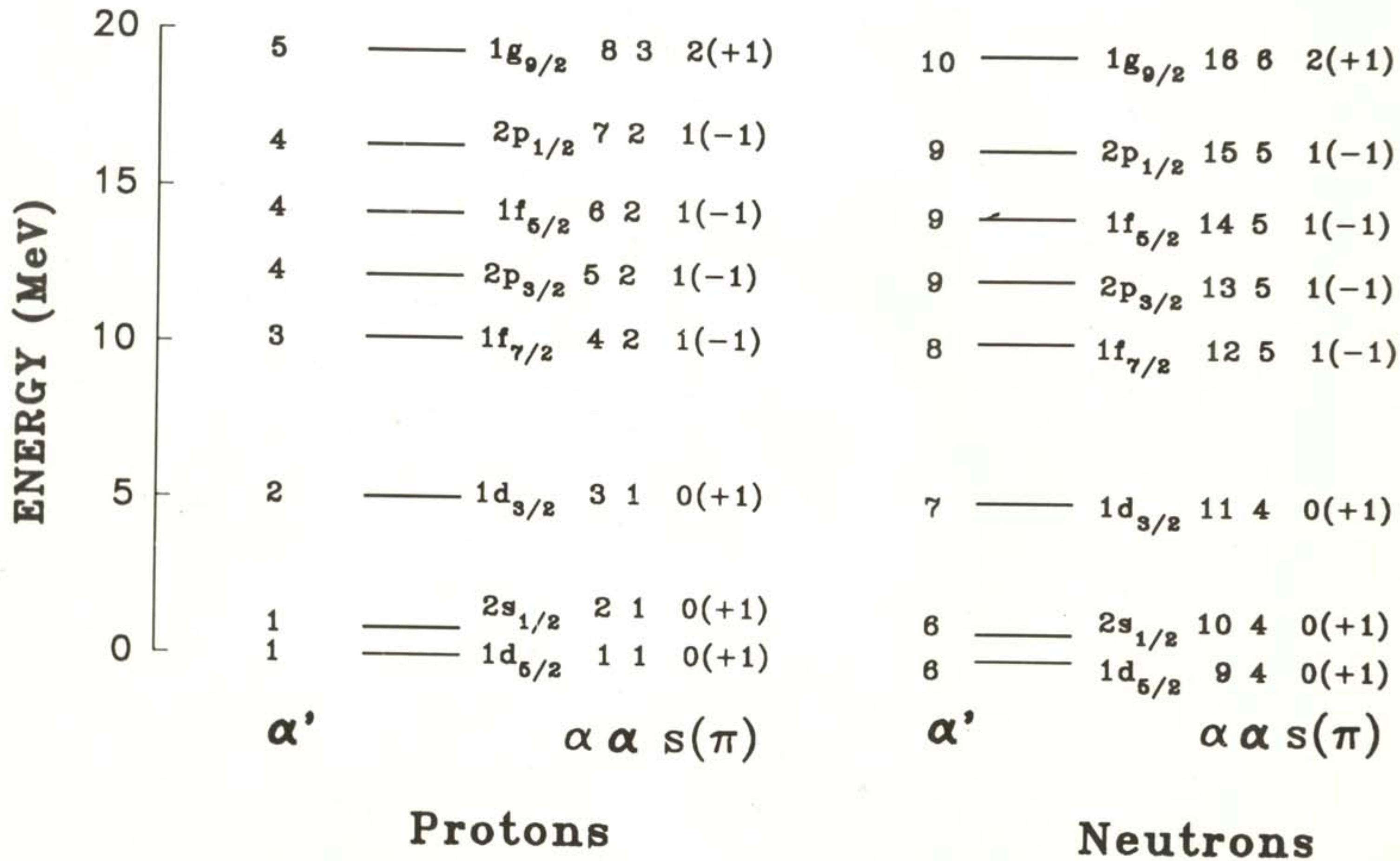
With spherical configurations **m** we have [J.B. French and V.K.B. Kota, Phys. Rev. Lett. **51**, 2183 (1983); Fig. 2],

$$\begin{aligned}
&\rho^{\mathbf{h}+\mathbf{V},m}(E) = \langle \delta(\mathbf{h}+\mathbf{V}-E)\rangle^{m} \\
&= d(m)^{-1} \sum_{\mathbf{m}} d(\mathbf{m})\, \langle \delta(\mathbf{h}+\mathbf{V}-E)\rangle^{\mathbf{m}} \\
&\to d(m)^{-1} \sum_{\mathbf{m}} d(\mathbf{m})\, \langle \delta(\epsilon(\mathbf{m})+\mathbf{V}-E)\rangle^{\mathbf{m}} \\
&= d(m)^{-1} \sum_{\mathbf{m}} d(\mathbf{m}) \int dz\, \langle \delta(\mathbf{h}-z)\rangle^{\mathbf{m}}\, \langle \delta(\mathbf{V}-E+z)\rangle^{\mathbf{m}} \\
&\to d(m)^{-1} \sum_{\mathbf{m}} d(\mathbf{m}) \rho^{\mathbf{h},\mathbf{m}} \otimes \rho^{\mathbf{V},m}[E] \\
&= \rho^{\mathbf{h},m} \otimes \rho^{\mathbf{V},m}[E]
\end{aligned}$$

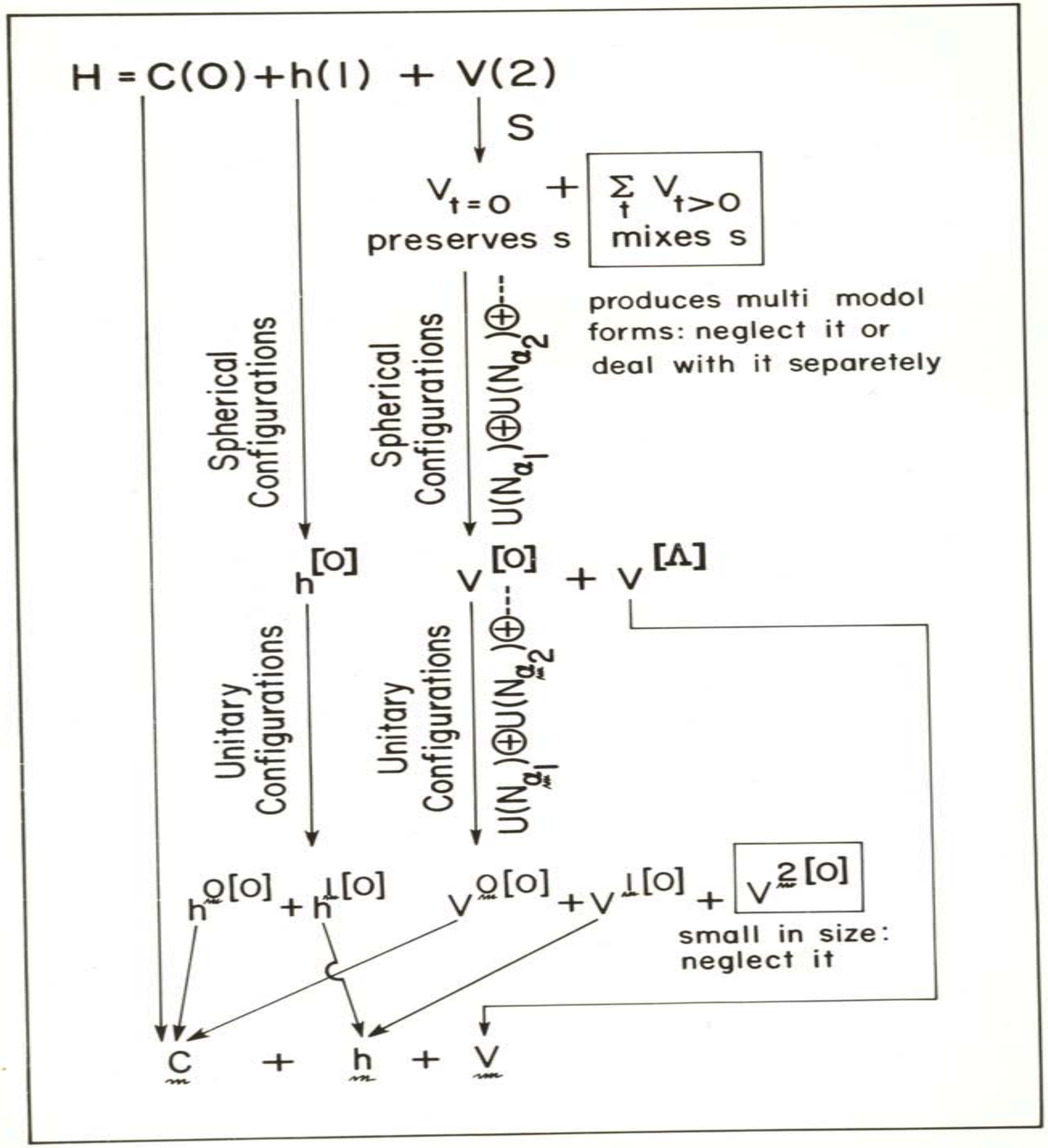
Here **h** is a algebraic mean-field hamiltonian. Using unitary orbits and the unitary decomposition [See Fig. 2; V.K.B. Kota and D. Majumdar, Nucl. Phys. **A604**, 129 (1996)] gives,

$$\begin{aligned}
I^{H}(E) &= \sum_{S} I^{\mathbf{h},S} \otimes \rho_{\mathcal{G}}^{\mathbf{V},S}[E] \\
&= \sum_{S} \left\{ \sum_{[\mathbf{m}]\in S} I^{\mathbf{h},[\mathbf{m}]} \otimes \rho_{\mathcal{G}}^{\mathbf{V},[\mathbf{m}]}[E] \right\} \\
\rho_{\mathcal{G}}^{\mathbf{V},[\mathbf{m}]} \Leftrightarrow \sigma_{\mathbf{V}}([\mathbf{m}]) &= \left\{ \langle \mathbf{V}^2 \rangle^{[\mathbf{m}]} \right\}^{1/2}
\end{aligned}$$

The $I^{\mathbf{h},[\mathbf{m}]}$ can be constructed as Edgeworth corrected Gaussians. Similarly $\sigma_{\mathbf{v}}([\mathbf{m}])$ can be calculated using trace propagation equations.



ENERGY (MeV)
20
15
10
5
0
5 1g9/2 8 3 2(+1)
4 2p1/2 7 2 1(−1)
4 1f5/2 6 2 1(−1)
4 2p3/2 5 2 1(−1)
3 1f7/2 4 2 1(−1)
2 1d3/2 3 1 0(+1)
1 2s1/2 2 1 0(+1)
1 1d5/2 1 1 0(+1)
α' α α s(π)
Protons
10 1g9/2 16 6 2(+1)
9 2p1/2 15 5 1(−1)
9 1f5/2 14 5 1(−1)
9 2p3/2 13 5 1(−1)
8 1f7/2 12 5 1(−1)
7 1d3/2 11 4 0(+1)
6 2s1/2 10 4 0(+1)
6 1d5/2 9 4 0(+1)
α' α α s(π)
Neutrons



H = C(0) + h(1) + V(2)
S
V t=0 + Σ t V t>0
preserves s
mixes s
produces multi modol forms: neglect it or deal with it separetely
Spherical Configurations
Spherical Configurations
U(N α1)⊕U(N α2)⊕---
h [0]
V [0] + V [Λ]
Unitary Configurations
Unitary Configurations
U(N α1)⊕U(N α2)⊕---
h 0[0] + h 1[0]
V 0[0] + V 1[0] + V 2[0]
small in size: neglect it
C + h + V

Spin-cutoff factors are calculated via spin-cutoff densities,

$$I^H_{J_Z^2}(E) = \langle J_Z^2\rangle^E I(E) = \langle\langle J_Z^2 \delta(H-E)\rangle\rangle^m$$
$$= \sum_S I^{\mathbf{h},S}_{J_Z^2} \otimes \rho^{\mathbf{V},S}_{J_Z^2:\mathcal{G}}[E]$$
$$= \sum_S \left\{ \sum_{[\mathbf{m}]\in S} I^{\mathbf{h},[\mathbf{m}]}_{J_Z^2} \otimes \rho^{\mathbf{V},[\mathbf{m}]}_{J_Z^2:\mathcal{G}}[E] \right\} ,$$
$$\rho^{\mathbf{V},[\mathbf{m}]}_{J_Z^2}(y) = \langle J_Z^2 \delta(\mathbf{V}-y)\rangle^{[\mathbf{m}]} / \langle J_Z^2\rangle^{[\mathbf{m}]} ;$$

$$\rho^{\mathbf{V},[\mathbf{m}]}_{J_Z^2:\mathcal{G}} \Leftrightarrow \begin{cases} \epsilon_{\mathbf{V}:J_Z^2}([\mathbf{m}]) = \langle J_Z^2 \mathbf{V}\rangle^{[\mathbf{m}]} / \langle J_Z^2\rangle^{[\mathbf{m}]} \\ \sigma_{\mathbf{V}:J_Z^2}([\mathbf{m}]) = \left[\langle J_Z^2 \mathbf{V}^2\rangle^{[\mathbf{m}]} / \langle J_Z^2\rangle^{[\mathbf{m}]} - \left\{\epsilon_{\mathbf{V}:J_Z^2}([\mathbf{m}])\right\}^2\right]^{1/2} \end{cases}$$

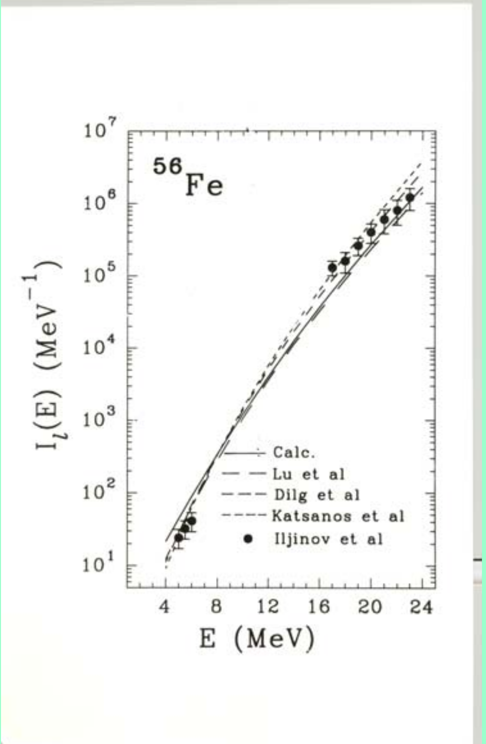
The $I^{\mathbf{h},[\mathbf{m}]}_{J_Z^2}$ follow from $I^{\mathbf{h},[\mathbf{m}]}$ easily by parametric differentiation.

One final problem is the determination of the ground state. The reference energy method [K.F. Ratcliff, Phys. Rev. C **3**, 117 (1971)] gives,

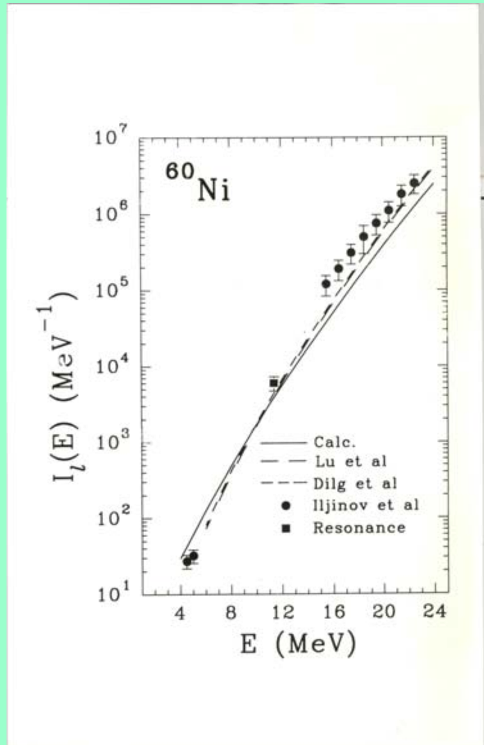
$$\left[N^{expt}_{ref} - \tfrac{1}{2}(2J^{expt}_{ref}+1)\right] = \sum_{[\mathbf{m}]} d([\mathbf{m}]) \int_{-\infty}^{\overline{E}_{ref}} \rho^{[\mathbf{m}]}(E)dE$$

$$E_{GS} = \overline{E}_{ref} - E^{expt}_{ref}$$

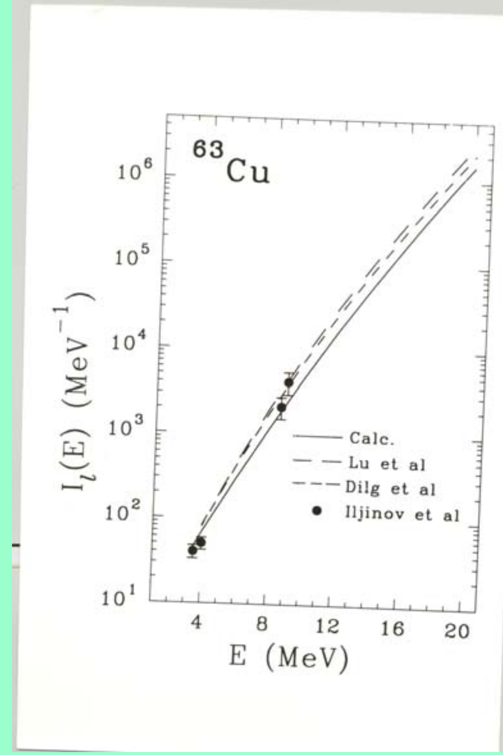
E^{expt}_{ref} is chosen to be as high in the spectrum as possible so that the non-chaotic low-lying levels are eliminated. This requires complete experimental spectrum up to E^{expt}_{ref}.



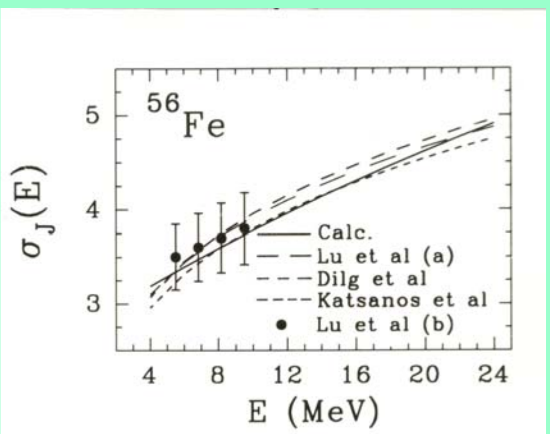
56Fe
I_l(E) (MeV^-1)
E (MeV)
Calc.
Lu et al
Dilg et al
Katsanos et al
Iljinov et al



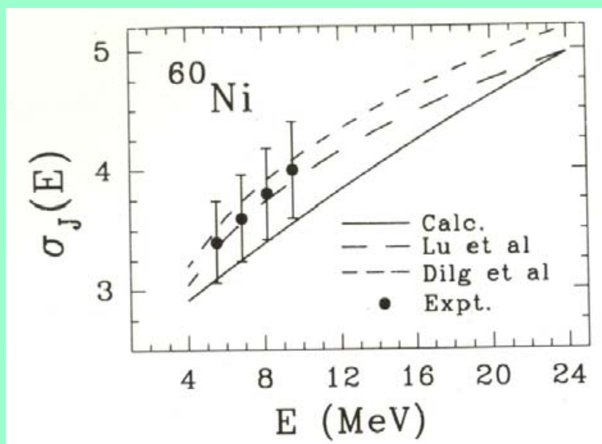
60Ni
I_l(E) (MeV^-1)
E (MeV)
Calc.
Lu et al
Dilg et al
Iljinov et al
Resonance



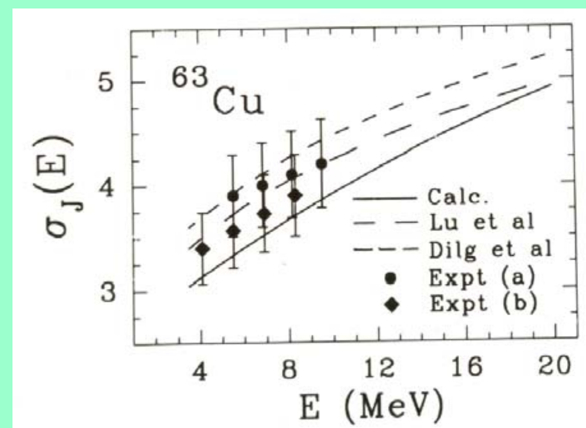
63Cu
I_l(E) (MeV^-1)
E (MeV)
Calc.
Lu et al
Dilg et al
Iljinov et al



56Fe
σ_J(E)
E (MeV)
Calc.
Lu et al (a)
Dilg et al
Katsanos et al
Lu et al (b)



60Ni
σ_J(E)
E (MeV)
Calc.
Lu et al
Dilg et al
Expt.



63Cu
σ_J(E)
E (MeV)
Calc.
Lu et al
Dilg et al
Expt (a)
Expt (b)

New advances in applying **SNSM** are due to Zelevinsky et al [M. Horoi, J. Kaiser, and V. Zelevinsky, Phys. Rev. C **67**, 054309 (2003); M. Horoi, M. Ghita, and V. Zelevinsky, Phys. Rev. C **69**, 041307(R) (2004), Nucl. Phys. **A758**, 142c (2005)].

They use:

1. a new exponential convergence method for fixing the ground state;
2. include a method for eliminating center of mass excitations (in multi-shell examples);
3. initially advocated that exact centroids and variances for fixed-mJT Gaussians can be calculated and used;
4. more recently accepted that J projection by spin-cutoff factors is the only practical method.

Q: How to deal with mixing between distant configurations (labeled by S)? We need results for partitioned EGOE – they will give multi-modal densities [V.K.B. Kota, D. Majumdar, R.U. Haq and R.J. Leclair, Can. J. Phys. 77, 893 (1999)].

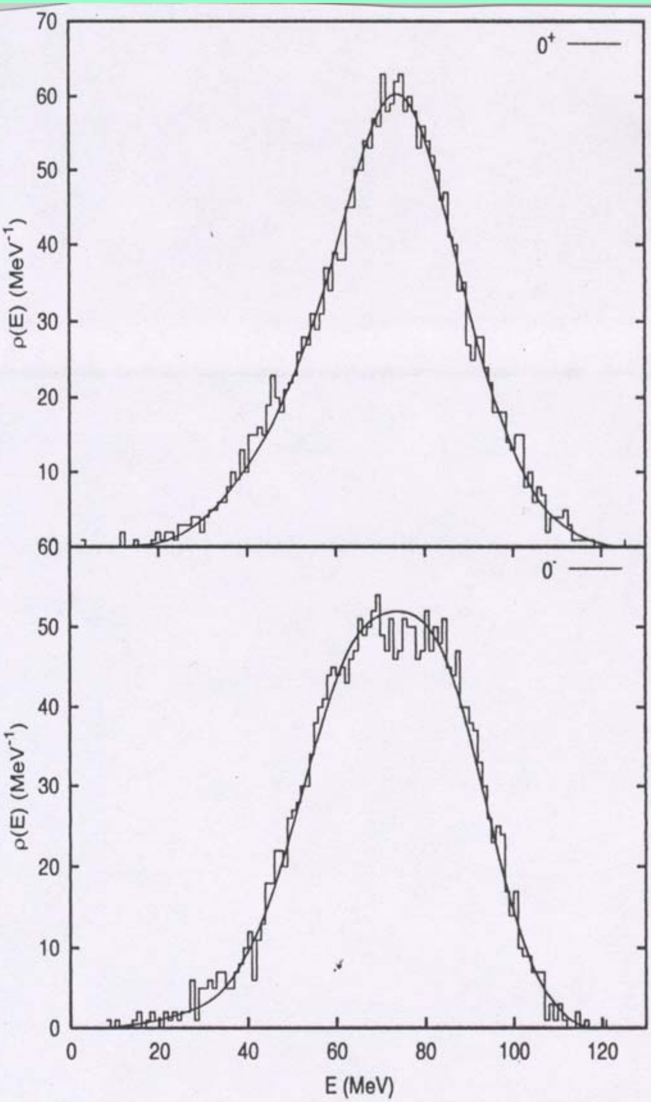


FIG. 6. $J=0$ shell model level density for 6 particles in *psd* model space (histograms) compared with the fixed-J sum of finite range Gaussians, Eq. (11); upper panel shows the density of positive parity states, while lower panel shows the negative parity states.

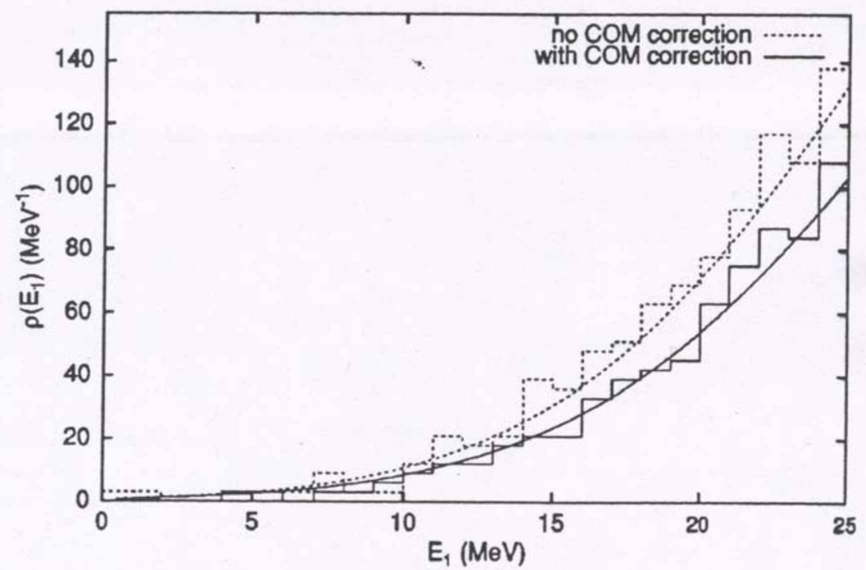


FIG. 1. Low-lying portion of the $J^{\pi}=1^{-}$ level density for 16 particles in the $p-sd$ shell. Only $1\hbar\omega$ configurations are included. COM indicates if the center-of-mass correction was taken into account (see text for details).

IV. SNSM FOR β-DECAY RATES: EXAMPLES

For applying SNSM for -decay rates, we need GT strength densities and they can be constructed using [J.B. French, V.K.B. Kota, A. Pandey, and S. Tomsovic, Ann. Phys. (N.Y.) **181**, 235 (1988); S. Tomsovic, Ph. D. Thesis, University of Rochester (1986) (unpublished); V.K.B. Kota and D. Majumdar, Z. Phys. A **351**, 377 (1995)],

$$I_{\mathcal{O}}^{H=\mathbf{h}+\mathbf{V}}(E_i, E_f) = \mathbf{I}_{\mathcal{O}}^{\mathbf{h}} \otimes \rho^{\mathbf{V}}_{\mathcal{O};BIV-\mathcal{G}}[E_i, E_f]$$

$$\Longrightarrow$$

$$\mathbf{I}_{\mathcal{O}(GT)}^{H=\mathbf{h}+\mathbf{V}}(E_i, E_f) = \sum_S \sum_{[\mathbf{m}_p^i,\mathbf{m}_n^i],[\mathbf{m}_p^f,\mathbf{m}_n^f]\in S}$$

$$\mathbf{I}_{\mathcal{O}(GT)}^{\mathbf{h};[\mathbf{m}_p^i,\mathbf{m}_n^i],[\mathbf{m}_p^f,\mathbf{m}_n^f]} \otimes \rho_{\mathcal{O}(GT)}^{\mathbf{V};[\mathbf{m}_p^i,\mathbf{m}_n^i],[\mathbf{m}_p^f,\mathbf{m}_n^f]}[E_i, E_f]\,;$$

$$\rho_{\mathcal{O}(GT)}^{\mathbf{V};[\mathbf{m}_p^i,\mathbf{m}_n^i],[\mathbf{m}_p^f,\mathbf{m}_n^f]}(x,y) =$$

$$\rho^{\mathbf{V}}_{\mathcal{O}(GT);BIV-\mathcal{G}}(x, y; 0, 0, \sigma_{\mathbf{V}}([\mathbf{m}_p^i, \mathbf{m}_n^i]), \sigma_{\mathbf{V}}([\mathbf{m}_p^f, \mathbf{m}_n^f]), \bar{\zeta})$$

$\mathbf{I}^{\mathbf{h}}$ is constructed as a bivariate Gaussian with Edgeworth corrections. The variances that define $\rho^{\mathbf{V}}$ involve approximations. Similarly the bivariate correlation coefficient is,

$$\bar{\zeta} \sim \langle \mathcal{O}^\dagger(GT)\, \mathbf{V}\, \mathcal{O}(GT)\, \mathbf{V} \rangle \,/\, \langle \mathcal{O}^\dagger(GT)\, \mathcal{O}(GT) \rangle \langle \mathbf{V}\mathbf{V} \rangle$$

β-DECAY RATES RELEVANT FOR PRESUPERNOVAE STARS

[I] A ~ 60-65, ^{61,62}Fe and $^{62\text{-}64}$Co: V.K.B. Kota and D. Majumdar, Z. Phys. A**351**, 377 (1995).

i. $1f_{7/2}$, $2p_{3/2}$, $1f_{5/2}$, $2p_{1/2}$ and $1g_{9/2}$ orbits with s = 0,0,0,0 and 1 respectively. SPE are -2.664 MeV, -0.644 MeV, 3.526 MeV and 1.366 MeV respectively for the *fp* orbits and $\Delta_{(fp)\,-g9/2}$ = 6MeV; for ^{64}Co it is taken to be 7 MeV.
ii. Calculations are performed in S = 0 ⊕1 ⊕ 2 spaces using *pn* unitary configurations with (*fp*) and $g_{9/2}$ orbits as unitary orbits.
iii. $I^{\mathbf{h}}$ is constructed using the above SPE.
iv. $\rho^{\mathbf{V}}$ is constructed by calculating the spreading variances using SDI with strength G = 20/A MeV; $\sigma_{\mathbf{V}}$ ~ 4.5 MeV for S=0 and ~ 6 MeV for S=2.
v. GS is fixed by demanding that the total level density at 8 MeV excitation is same as the Fermi gas value with (a, Δ) given by Dilg et al formula - for the nuclei under consideration, completeness of the low-energy spectrum is not known.
vi. The S = 2 intensities in the GS domain are ≤ 30% of the S = 0 intensities - thus the $g_{9/2}$ orbit is seen to be important.
vii. the GT (β^-) NEWSR as predicted by the present calculations are compared with shell model results: for ^{54}Fe, ^{56}Fe, ^{60}Fe, ^{58}Ni and ^{60}Ni they are (17.8, 22.6, 31.6, 20.2, 24.2) and (15.1, 22.1, 33.5, 16.6, 24.6) from **Calc** and **SM** respectively.
viii. Assuming the EGOE form of $\overline{\zeta}$, $\zeta(m) = \zeta_0 + \zeta_1 / m$ where m is the number of valence nucleons/holes and minimizing the RMS of $\log(\tau^i_{1/2})$ between calc and expt for all the nuclei, $\overline{\zeta}$ for each nucleus is determined; $\overline{\zeta}$: 0.67
ix. As $\overline{\zeta}$ decreases the calculated half life increases. Similarly the β^- decay rates go down with decreasing $\overline{\zeta}$ . .

Nucleus	Q (MeV)	$\bar{\zeta}$	half life (s) Calc	Expt
^{64}Co	7.307	0.668	1.9	0.3
^{63}Co	3.662	0.671	52.7	27.5
^{62}Co	5.315	0.675	16.5	90
^{62}Fe	2.327	0.675	267.2	68
^{61}Fe	3.890	0.680	23.0	360

Nucleus	ρ(gms/cc)	Y_e	Temperature (^oK) 3×10^9	4×10^9	5×10^9
			β^--decay rate (s^{-1})		
^{64}Co	10^9	0.50	0.85×10^{-1}	1.04×10^{-1}	1.30×10^{-1}
		0.47	0.91×10^{-1}	1.11×10^{-1}	1.38×10^{-1}
		0.43	1.00×10^{-1}	1.21×10^{-1}	1.49×10^{-1}
	10^8	0.50	3.10×10^{-1}	3.43×10^{-1}	3.84×10^{-1}
		0.47	3.14×10^{-1}	3.47×10^{-1}	3.87×10^{-1}
		0.43	3.19×10^{-1}	3.52×10^{-1}	3.93×10^{-1}
	10^7	0.50	3.83×10^{-1}	4.14×10^{-1}	4.54×10^{-1}
		0.47	3.83×10^{-1}	4.14×10^{-1}	4.54×10^{-1}
		0.43	3.84×10^{-1}	4.15×10^{-1}	4.55×10^{-1}
^{63}Co	10^9	0.50	0.12×10^{-3}	0.65×10^{-3}	2.26×10^{-3}
		0.47	0.15×10^{-3}	0.76×10^{-3}	2.54×10^{-3}
		0.43	0.21×10^{-3}	0.94×10^{-3}	2.98×10^{-3}
	10^8	0.50	0.85×10^{-2}	1.31×10^{-2}	2.09×10^{-2}
		0.47	0.88×10^{-2}	1.35×10^{-2}	2.14×10^{-2}
		0.43	0.93×10^{-2}	1.41×10^{-2}	2.21×10^{-2}
	10^7	0.50	1.63×10^{-2}	2.17×10^{-2}	3.06×10^{-2}
		0.47	1.64×10^{-2}	2.18×10^{-2}	3.07×10^{-2}
		0.43	1.65×10^{-2}	2.19×10^{-2}	3.08×10^{-2}
^{62}Fe	10^9	0.50	0.04×10^{-4}	1.16×10^{-4}	9.99×10^{-4}
		0.47	0.06×10^{-4}	1.42×10^{-4}	11.50×10^{-4}
		0.43	0.09×10^{-4}	1.88×10^{-4}	14.00×10^{-4}
	10^8	0.50	2.40×10^{-3}	6.56×10^{-3}	1.62×10^{-2}
		0.47	2.57×10^{-3}	6.86×10^{-3}	1.67×10^{-2}
		0.43	2.83×10^{-3}	7.29×10^{-3}	1.73×10^{-2}
	10^7	0.50	8.11×10^{-3}	1.43×10^{-2}	2.71×10^{-2}
		0.47	8.19×10^{-3}	1.44×10^{-2}	2.72×10^{-2}
		0.43	8.30×10^{-3}	1.46×10^{-2}	2.74×10^{-3}

[II] A ~ 65-75: D. Majumdar and K. Kar, preprint astro-ph/0205218 (2002); private communication.

(i) Most of the nuclei studied here appear among the top 70 nuclei for β^- decay as given in M.B. Aufderheide et al, Ap. J. Supp **91**, 389 (1994). Some examples are: $^{67-69,71}$Ni, ^{66,68,69}Co, ^{68,72,74}Cu.

(ii) $1f_{7/2}$, $2p_{3/2}$, $1f_{5/2}$, $2p_{1/2}$, $1g_{9/2}$, $2d_{5/2}$,$1g_{7/2}$, $3s_{1/2}$, $2d_{3/2}$ with SPE (in MeV) taken as 24.5, 26.58, 26.19, 29.09, 33.91, 38.52, 42.47, 42.30, 43.15 respectively (Seeger energies).

(iii) Calculations are performed in S = 0 $\oplus$ 1 $\oplus$ 2 spaces using *pn* unitary configurations with (*fp*), $1g_{9/2}$ and ($2d_{5/2}$,$1g_{7/2}$, $3s_{1/2}$, $2d_{3/2}$) orbits as unitary orbits with s=0,1 and 2.

(iv) Pairing plus Q.Q interaction with strength χ=242/A$^{5/3}$ is used; $\langle \mathbf{V}^2 \rangle \sim 15.5$ MeV^2.

(v) Level densities are calculated using Dilg et al formula.

(vi) GS is fixed just as in the case of A ~ 60-65 nuclei.

(vii) $\overline{\zeta}$ for each nucleus is determined just as before; $\overline{\zeta} \sim 0.66$.

(viii) For neutron rich nuclei ($T > 2$), calculations with fixed (m_p, m_n) spaces without T projection is a reasonable procedure - this is checked.

(ix) In the calculation of half lives and rates, low-lying log *ft*'s wherever known are incorporated with the total strength suitably adjusted.Thus the known expt'l information is used to make the rates more realistic.

(x) These calculations are extended to EC rates for 65 < A < 110 nuclei.

Nucleus	ρ(gms/cc)	Y_e	Temperature in °K		
			3×10^9	4×10^9	5×10^9
			Rates (s^{-1})		
^{65}Ge	10^9	0.50	2.77×10^{-3}	2.89×10^{-3}	3.07×10^{-3}
		0.47	2.56×10^{-3}	2.68×10^{-3}	2.86×10^{-3}
		0.43	2.28×10^{-3}	2.41×10^{-3}	2.58×10^{-3}
	10^8	0.50	2.05×10^{-4}	2.47×10^{-4}	3.13×10^{-4}
		0.47	1.92×10^{-4}	2.34×10^{-4}	2.98×10^{-4}
		0.43	1.76×10^{-4}	2.16×10^{-4}	2.79×10^{-4}
	10^7	0.50	2.96×10^{-5}	5.67×10^{-5}	1.10×10^{-4}
		0.47	2.86×10^{-5}	5.56×10^{-5}	1.10×10^{-4}
		0.43	2.71×10^{-5}	5.43×10^{-5}	1.09×10^{-4}
^{69}Se	10^9	0.50	2.81×10^{-3}	2.92×10^{-3}	3.10×10^{-3}
		0.47	2.60×10^{-3}	2.72×10^{-3}	2.89×10^{-3}
		0.43	2.32×10^{-3}	2.45×10^{-3}	2.62×10^{-3}
	10^8	0.50	2.20×10^{-4}	2.64×10^{-4}	3.31×10^{-4}
		0.47	2.07×10^{-4}	2.50×10^{-4}	3.16×10^{-4}
		0.43	1.89×10^{-4}	2.31×10^{-4}	2.96×10^{-4}
	10^7	0.50	3.26×10^{-5}	6.16×10^{-5}	1.19×10^{-4}
		0.47	3.14×10^{-5}	6.05×10^{-5}	1.18×10^{-4}
		0.43	2.99×10^{-5}	5.90×10^{-5}	1.17×10^{-4}
^{73}Kr	10^9	0.50	2.18×10^{-3}	2.27×10^{-3}	2.40×10^{-3}
		0.47	2.01×10^{-3}	2.11×10^{-3}	2.24×10^{-3}
		0.43	1.80×10^{-3}	1.90×10^{-3}	2.03×10^{-3}
	10^8	0.50	1.72×10^{-4}	2.05×10^{-4}	2.58×10^{-4}
		0.47	1.61×10^{-4}	1.95×10^{-4}	2.46×10^{-4}
		0.43	1.48×10^{-4}	1.80×10^{-4}	2.30×10^{-4}
	10^7	0.50	2.55×10^{-5}	4.82×10^{-5}	9.24×10^{-5}
		0.47	2.46×10^{-5}	4.73×10^{-5}	9.19×10^{-5}
		0.43	2.34×10^{-5}	4.61×10^{-5}	9.13×10^{-5}

Electron capture rates

Nucleus	ρ(gms/cc)	Y_e	Temperature in °K		
			3×10^9	4×10^9	5×10^9
			Rates (s^{-1})		
^{77}Sr	10^9	0.50	1.86×10^{-3}	1.90×10^{-3}	1.98×10^{-3}
		0.47	1.72×10^{-3}	1.77×10^{-3}	1.86×10^{-3}
		0.43	1.54×10^{-3}	1.60×10^{-3}	1.68×10^{-3}
	10^8	0.50	1.56×10^{-4}	1.84×10^{-4}	2.26×10^{-4}
		0.47	1.47×10^{-4}	1.74×10^{-4}	2.16×10^{-4}
		0.43	1.35×10^{-4}	1.62×10^{-4}	2.03×10^{-4}
	10^7	0.50	2.39×10^{-5}	4.41×10^{-5}	8.27×10^{-5}
		0.47	2.30×10^{-5}	4.33×10^{-5}	8.23×10^{-5}
		0.43	2.19×10^{-5}	4.22×10^{-5}	8.18×10^{-5}
^{81}Zr	10^9	0.50	1.45×10^{-3}	1.48×10^{-3}	1.55×10^{-3}
		0.47	1.34×10^{-3}	1.38×10^{-3}	1.44×10^{-3}
		0.43	1.20×10^{-3}	1.24×10^{-3}	1.31×10^{-3}
	10^8	0.50	1.15×10^{-4}	1.36×10^{-4}	1.68×10^{-4}
		0.47	1.09×10^{-4}	1.29×10^{-4}	1.60×10^{-4}
		0.43	9.96×10^{-5}	1.19×10^{-4}	1.50×10^{-4}
	10^7	0.50	1.72×10^{-5}	3.20×10^{-5}	6.04×10^{-5}
		0.47	1.66×10^{-5}	3.14×10^{-5}	6.01×10^{-5}
		0.43	1.58×10^{-5}	3.06×10^{-5}	5.97×10^{-5}

Sr No.	Nucl.	Z	Q_{val} (MeV)	$T_{1/2}^{expt.}$ sec.	$T_{1/2}^{calc}$ sec.	ζ
1	^{66}Co	27	10.0	0.23	0.32	0.6627
2	^{67}Ni	28	3.385	21.0	26.91	0.6603
3	^{68}Co	27	9.30	0.18	0.20	0.6581
4	^{68}Ni	28	2.06	19.0	18.50	0.6581
5	^{68}Cu	29	4.46	31.1	31.23	0.6581
6	^{69}Co	27	9.30	0.27	0.36	0.6561
7	^{69}Ni	28	5.36	11.4	12.04	0.6561
8	^{71}Ni	28	6.90	1.86	1.62	0.6524
9	^{72}Cu	29	8.22	6.60	6.13	0.6507
10	^{74}Cu	29	9.99	1.59	0.40	0.6507

Table 1: β^- decay half lives and comparison with experiment values

Nucleus	ρ(gms/cc)	Y_e	Temperature in °K			
			2×10^9	3×10^9	4×10^9	5×10^9
			Rates (s^{-1})			
^{69}Co	10^9	0.47	7.25×10^{-3}	1.18×10^{-1}	2.96×10^{-1}	3.63×10^{-1}
	10^8	0.47	1.64×10^{-2}	2.62×10^{-1}	6.37×10^{-1}	7.53×10^{-1}
	10^7	0.47	1.94×10^{-2}	3.05×10^{-1}	7.32×10^{-1}	8.55×10^{-1}
	10^9	0.45	7.47×10^{-3}	1.22×10^{-1}	3.05×10^{-1}	3.73×10^{-1}
	10^8	0.45	1.66×10^{-2}	2.64×10^{-1}	6.41×10^{-1}	7.57×10^{-1}
	10^7	0.45	1.94×10^{-2}	3.06×10^{-1}	7.33×10^{-1}	8.56×10^{-1}
^{68}Ni	10^9	0.47	8.38×10^{-6}	7.81×10^{-4}	6.00×10^{-3}	1.59×10^{-2}
	10^8	0.47	2.18×10^{-3}	4.38×10^{-2}	1.34×10^{-1}	1.93×10^{-1}
	10^7	0.47	4.55×10^{-3}	8.08×10^{-2}	2.20×10^{-1}	2.90×10^{-1}
	10^9	0.45	1.08×10^{-5}	9.18×10^{-4}	6.76×10^{-3}	1.75×10^{-2}
	10^8	0.45	2.25×10^{-3}	4.50×10^{-2}	1.36×10^{-1}	1.97×10^{-1}
	10^7	0.45	4.57×10^{-3}	8.11×10^{-2}	2.20×10^{-1}	2.91×10^{-1}

Table 2: β^- decay rates for densities and temperatures relevant for supernova core

β-DECAY RATES FOR r-PROCESS NUCLEOSYNTHESIS

Instead of using the SNSM from first principles, it is possible to combine this method with experimental data and phenomenological formulas wherever known - the resulting hybrid approach is closer to FFN method. One example for this is given here: K. Kar, S. Chakravarti and V.R. Manfredi, to be published.

(i) Rates are calculated for nuclei near the N=82 magic shell with 115 < A <140.

(ii) Used low-lying states for which the log *ft*'s are known experimentally.

(iii) Fermi strength goes to IAS and the spreading of this strength is due to Coulomb interaction. The width is $\sigma_c \sim 0.157\ Z\ A^{-1/3}$ MeV. As the width is small, it can not be reached by the Q-value. Therefore Fermi transitions make little contribution.

(iv) With $\rho(E_i,E_f)=\rho_{21}(E_f|E_i)\ \rho_1(E_i)$, it is easy to see that $\rho_1(E_i)$ is GT strength sum. Formula or theory for strength sums is used. Then it is assumed (follows from a single bivariate Gaussian form for GT transition strength densities), that $\rho_{21}(E_f|E_i)$ is a Gaussian.

(v) Formula, with $E_i=E_{GS}$ [see for example Pruet and G.M. Fuller, APJS **149**, 189 (2003)], for the centroid $\epsilon(E_{GS})$ of $\rho_{21}(E_f|E_i= E_{GS})$ is used. Brink's hypothesis gives $\epsilon(E_i) = \epsilon(E_{GS}) +(E_i - E_{GS})$. The structure here is similar to the conditional centroid of a bivariate Gaussian.

(vi) The width of $\rho_{21}(E_f|E_i)$ (independent of E_i) is treated as a free parameter and determined via best fit for half lives; $\sigma \sim 5$ MeV.

(vii) In the first calculations only GS of the mother nucleus are considered.

	Mother nucleus	Daughter nucleus	Q-value (MeV)	No of lowlying log ft's taken	$\tau_{1/2}$ Exp (sec)	$\tau_{1/2}$ Calc (sec)
1.	$^{138}_{53}I_{85}$	^{138}Xe	7.820	3	6.49	12.99
2.	$^{137}_{53}I_{84}$	^{137}Xe	5.880	2	24.5	33.1
3.	$^{136}_{53}I_{83}$	^{136}Xe	6.930	3	83.4	69.5
4.	$^{134}_{51}Sb_{83}$	^{134}Te	8.420	2	10.43	7.72
5.	$^{133}_{50}Sn_{83}$	^{133}Sb	7.830	4	1.44	1.19
6.	$^{132}_{51}Sb_{81}$	^{132}Te	5.290	2	167.4	141.3
7.	$^{131}_{49}In_{82}$	^{131}Sn	6.746	2	0.282	0.280
8.	$^{130}_{49}In_{81}$	^{130}Sn	10.250	3	0.32	0.25
9.	$^{128}_{49}In_{79}$	^{128}Sn	8.980	3	0.84	2.30
10.	$^{125}_{48}Cd_{77}$	^{125}In	7.160	3	0.65	2.12
11.	$^{120}_{47}Ag_{73}$	^{120}Cd	8.200	4	1.23	0.987
12.	$^{118}_{47}Ag_{71}$	^{118}Cd	7.060	3	3.76	7.23
13.	$^{116}_{45}Rh_{71}$	^{116}Pd	8.900	3	0.68	0.49

Half lives of nuclei for which log *ft* values are available.

NUCLEUS	DENSITY (g/cm^3)	TEMPERATURE T_9 (in 10^9 K)			
		0.5	1.0	2.0	3.0
^{133}Sn	10^8	0.543	0.542	0.493	0.450
	10^7	0.581	0.558	0.498	0.452
	10^6	0.584	0.559	0.498	0.452
	10^5	0.584	0.560	0.498	0.452
	10^4	0.584	0.560	0.498	0.452
	10^3	0.584	0.560	0.498	0.452
^{132}In	10^8	1.70	1.75	1.59	1.46
	10^7	1.93	1.83	1.61	1.47
	10^6	1.95	1.83	1.61	1.47
	10^5	1.95	1.84	1.61	1.47
	10^4	1.95	1.84	1.61	1.47
	10^3	1.95	1.84	1.61	1.47

β-decay rates in s^{-1}

A SIMPLER APPROACH FOR GT MATRIX ELEMENTS

A new approach is to use a formula in terms of occupation numbers, state densities and a t-distribution [V.K.B. Kota and R. Sahu, Phys. Rev E **62**, 3568 (2000); V.K.B. Kota, N.D. Chavda and R. Sahu, preprint **nlin.CD/0508023**]. Then,

$$\rho_{biv-t}(E_i, E_f; \epsilon_i, \epsilon_f, \sigma_1, \sigma_2, \zeta; \nu) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\zeta^2}} \times$$

$$\left[1 + \frac{1}{\nu(1-\zeta^2)}\left\{\left(\frac{E_i-\epsilon_i}{\sigma_1}\right)^2 - 2\zeta\left(\frac{E_i-\epsilon_i}{\sigma_1}\right)\left(\frac{E_f-\epsilon_f}{\sigma_2}\right) + \left(\frac{E_f-\epsilon_f}{\sigma_2}\right)^2\right\}\right]^{-\frac{\nu+2}{2}} \quad , \nu \geq 1 .$$

$$|\langle E_f \mid \mathcal{O} \mid E_i\rangle|^2 = \sum_{\alpha,\beta} |\epsilon_{\alpha\beta}|^2 \langle n_\beta(1-n_\alpha)\rangle^{E_i} \overline{D(E_f)} \; \mathcal{F} ;$$

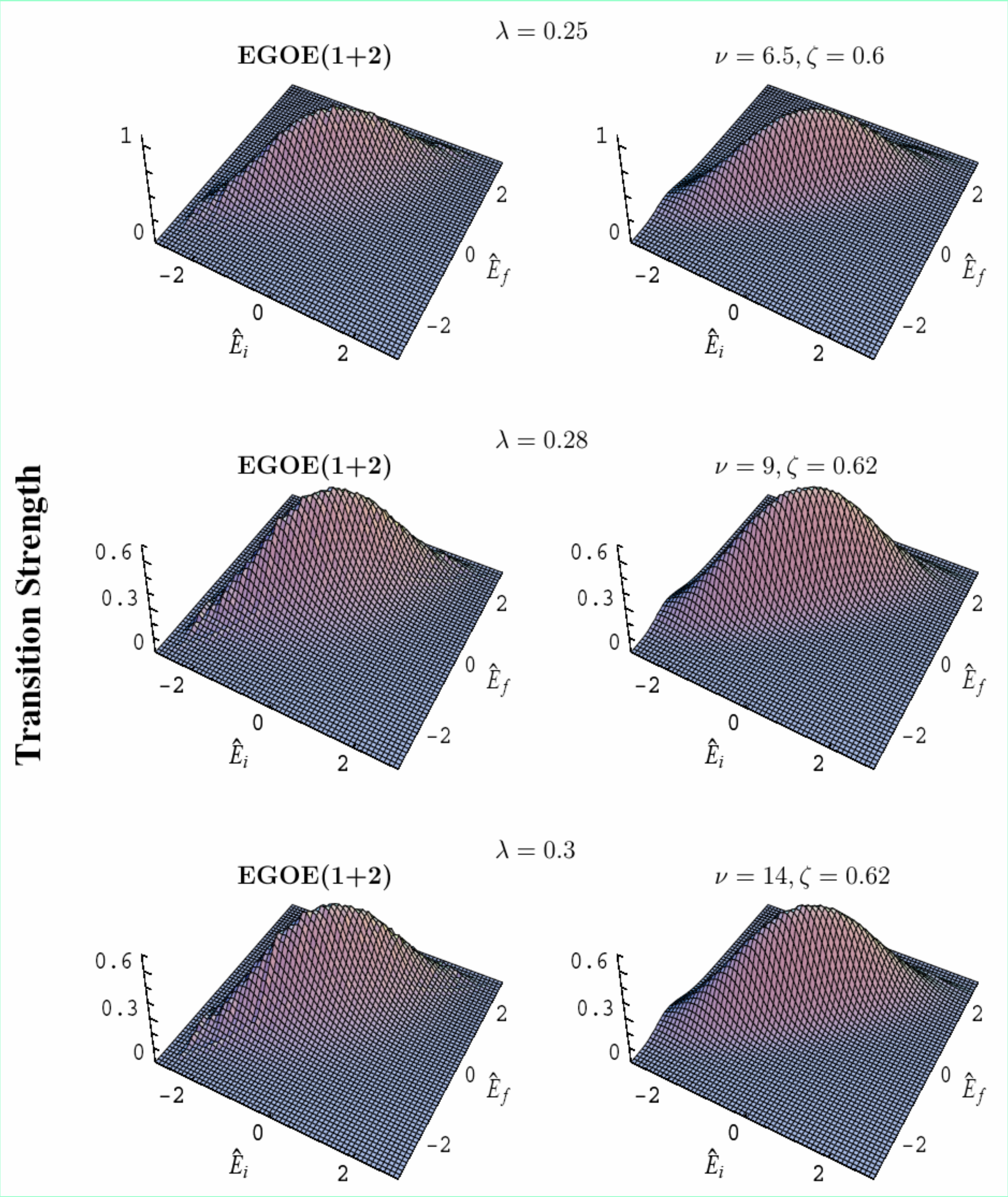
$$\mathcal{F} = \int_{-\infty}^{+\infty} \rho_{biv-t;\mathcal{O}}(E_i,\ E_f;\ \mathcal{E}_i,\ \mathcal{E}_f = \mathcal{E}_i - \epsilon_\beta + \epsilon_\alpha,\ \sigma_1,\ \sigma_2,\ \zeta;\nu)\, d\mathcal{E}_i$$

$$= \frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\pi}\Gamma(\frac{\nu}{2})} \frac{1}{\sqrt{\nu(\sigma_1^2+\sigma_2^2-2\zeta\sigma_1\sigma_2)}} \left[1 + \frac{\Delta^2}{\nu(\sigma_1^2+\sigma_2^2-2\zeta\sigma_1\sigma_2)}\right]^{-\frac{\nu+1}{2}} \; ; \; \Delta = E_f - E_i + \epsilon_\beta - \epsilon_\alpha$$

One can use the approximation $\langle n_\beta(1-n_\alpha)\rangle^{E_i} \approx \langle n_\beta\rangle^{E_i} \langle(1-n_\alpha)\rangle^{E_i}$ where $\langle n_\alpha\rangle^{E_i}$ are occupancies; here α are single particle states.

This formalism is closely related to the approach of FFN that is used recently in [Pruet and G.M. Fuller, APJS **149**, 189 (2003)].

***N*=12, *m*=6**



Transition Strength
λ = 0.25
EGOE(1+2)
ν = 6.5, ζ = 0.6
λ = 0.28
EGOE(1+2)
ν = 9, ζ = 0.62
λ = 0.3
EGOE(1+2)
ν = 14, ζ = 0.62
Ê_i
Ê_f

Thank you all